STABLE ∞-CATEGORIES AND ALGEBRAIC SURGERY

THOMAS NIKOLAUS

Abstract. These are notes for a lecture course given at the University of Münster in summer 2020. The course is mostly based on results obtained in joint work with Baptiste Calmes, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin and Wolfgang Steimle.

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1. Introduction

In this first section we want to give a rough overview of the main results that we will obtain in this course. We will say very little about the techniques and the reader not familiar with the basic objects should not be irritated as we will introduce most of them in this course. We do however assume that everyone is familiar with the notion of a spectrum, an ∞-category and how to work in the ∞-category of spectra. Most of the new results and ideas in this course are joint with Baptiste Calmes, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin and Wolfgang Steimle. We will refer to the collection of those authors as #nine.

1.1. **Statement of results.** Let R be a commutative ring. We want to study unimodular symmetric forms over R, i.e. symmetric bilinear forms

$$q : P \otimes_R P \to R$$

where P is a finitely generated, projective R-module and such that the adjunct map

$$\tilde{q} : P \to DP = \mathrm{Hom}_R(P, R) \qquad p \mapsto q(p, -)$$

is an isomorphism of R-modules. Two such forms (P, q) and (P', q') can be added and this equips the set of isomorphism classes (under the obvious notion of morphism) with the structure of an abelian monoid.

Definition 1.1. *The* (symmetric) Grothendieck Witt group *of a commutative ring R is defined as the group completion of the monoid of isomorphism classes of unimodular, symmetric forms:*

$$\mathrm{GW}_0^s(R) = \{\textit{Iso. classes of unimodular, symmetric forms over } R\}^{\mathrm{grp}} .$$

The (symmetric) Witt group[1] *of R is defined as*

$$\mathrm{W}_0^s(R) = \frac{\{\textit{Iso. classes of unimodular, symmetric forms over } R\}}{\{\textit{metabolic forms}\}}$$

where a form (P, q) is called metabolic *if it admits a* Lagrangian*, that is a projective, finitely generated submodule $i : L \subseteq P$ such that $q|_L = 0$ and such that the induced sequence*

$$0 \to L \xrightarrow{i} P \cong DP \xrightarrow{Di} DL \to 0$$

is short exact.

Note that the quotient in the Definition of the Witt group is taken in the category of abelian monoids, but we have the following result:

Proposition 1.2. *The abelian monoid $\mathrm{W}_0(R)$ is a group. Moreover the sequence*

$$\mathrm{K}_0(R) \xrightarrow{\mathrm{hyp}} \mathrm{GW}_0^s(R) \to W_0^s(R) \to 0$$

is exact. Here the map hyp *sends a class $[P] \in \mathrm{K}_0(R)$ represented by a projective module to the* hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *on the module $P \oplus DP$.*

The first part of the Lemma is straighforward to verify (and a recommendet exercise for the reader). For the second part of the Lemma one has to work a bit to see that after group completion hyperbolic and metabolic forms agree. Over general ring there is a big difference between hyperbolic forms and metabolic forms[2], but the statement asserts that after group completion this difference goes away.

Example 1.3. For the ring $R = \mathbb{Z}$ we have the sequence

$$\begin{tikzcd}
\mathrm{K}_0\,\mathbb{Z} \arrow[r] \arrow[d, "\simeq"', "\mathrm{rk}"] & \mathrm{GW}_0^s\,\mathbb{Z} \arrow[r] \arrow[d, "\simeq"', "(\mathrm{sgn}, \frac{\mathrm{rk}-\mathrm{sgn}}{2})"] & \mathrm{W}_0^s(\mathbb{Z}) \arrow[r] \arrow[d, "\simeq"', "\mathrm{sgn}"] & 0 \\
\mathbb{Z} \arrow[r, "i_2"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "p_1"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

[1] This should not be confused with the totally unrelated ring of Witt vectors $W(R)$. Also usually the Witt group is just denoted $W(R)$ instead of our $W_0(R)$.

[2] An example of a form over $\mathbb{F}_2$ that is metabolic but not hyperbolic is $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

where rk denotes the rank and sgn denotes the signature. The middle isomorphism is non-trivial and can be deduced from the classification of indefinite forms over $\mathbb{Z}$ as one can always make a form indefinite in the Grothendieck–Witt group by adding indefinite forms.

The main result that we want to explain in this lecture course is higher version of Proposition 1.2 and the computation of $\mathrm{GW}_0^s(\mathbb{Z})$ in Example 1.3. To formulate this we have to introduce the appropriate higher Grothendieck–Witt and Witt groups. We will assume that the reader is familiar with the Definiton of algebraic K-theory groups. This will also be recalled in the course.

Definition 1.4. *The higher (symmetric) Grothendieck–Witt groups* $\mathrm{GW}_n^s(R)$ *of* R *for* $n \geq 0$ *are the higher* K*-groups of the category of unimodular symmetric forms over* R*. These are the homotopy groups of the (symmetric) Grothendieck–Witt spectrum* $\mathrm{GW}^s(R)$ *defined as the* K*-theory spectrum of this category, i.e.*

$$\mathrm{GW}^s(R) = \{\textit{category of unimodular, symmetric forms over } R\}^{\mathrm{grp}} .$$

These higher Grothendieck–Witt groups are the groups that we are mainly interested in and which are generally very hard to compute. For example the case of the integers $\mathbb{Z}$ was so far completely open and will be resolved as one of the main goals of this lecture course. We will explain this below, but let us first state the first main result, which is our higher version of Propositon 1.2.

Theorem 1.5 (#nine)**.** *For every commutative ring* R *there is a fibre sequence of spectra*

$$K(R)_{hC_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}^s(R) \to \mathrm{L}^s_{\mathrm{cl}}(R) .$$

Here the (connective) K*-theory spectrum* $K(R)$ *carries the* C_2*-action given by sending a finite projective module* P *to its dual* DP*. The spectrum* $\mathrm{L}^s_{\mathrm{cl}}(R)$ *is a spectrum whose homotopy groups are Ranicki's classical symmetric* L*-theory groups.*

Let us explain the classical L-theory spectrum $\mathrm{L}^s_{\mathrm{cl}}(R)$ a bit more. It is connective and its homotopy groups are the L-theory groups $\pi_n(\mathrm{L}^s_{\mathrm{cl}}(R)) = \mathrm{L}^s_{\mathrm{cl},n}(R)$ which can be described as higher analogues of the Witt group. The idea is to replace projective modules by perfect chain complexes over R. More precisely we consider perfect chain complexes X with Tor-amplitude in $[-n, 0]$, i.e. those chain complexes that can up to quasi-isomorphism be represented by a chain complex of finitely generated, projective modules over R concentrated in homological degrees $[-n, 0]$ and vanishing outside of that range. There is a notion of symmetric unimodular forms and metabolic forms in this setting: a form on such a X is given by a map $(X \otimes X)_{hC_2} \to R[-n]$ and the unimodularity condition is that the associated map $X[n] \to DX$ is an equivalence. The condition of being metabolic will be explained below. We will refer to such objects as strictly n-dimensional unimodular forms (and strictly n-dimensional metabolic forms). Then we have

$$\mathrm{L}^s_{\mathrm{cl},n}(R) = \frac{\{\text{strictly } n\text{-dimensional unimodular, symmetric forms over } R\}}{\{\text{strictly } n\text{-dimensional metabolic forms}\}} .$$

In particular we have $\mathrm{L}^s_{\mathrm{cl},0}(R) = W_0(R)$ and $\mathrm{L}^s_{\mathrm{cl},1}(R)$ can also be descibed in terms of very explict algebraic objects (so-called symmetric formations). The higher L-groups are a priori much less accessible. But it turns out that these groups are 2-periodic (up to signs) if either $\frac{1}{2} \in R$ or R is a Dedekind domain. As a result, the higher L-groups are also very accessible and computable. Even without these assumptions on the ring R the L-theory is much easier to compute than K-theory and GW-theory, mostly due to extensive and impressive work by Ranicki.

Remark 1.6. Theorem 1.5 was essentially known in the case $\frac{1}{2} \in R$ by work of Karoubi, Schlichting and Hesselholt–Madsen. Upon taking π_0 it gives rise to an exact sequence

$$K_0(R)_{C_2} \to \mathrm{GW}_0^s(R) \to \mathrm{W}_0^s(R) \to 0$$

which differs from the sequence in Proposition 1.2 only by the C_2-orbits, but the exactness is equivalent.

We also note that there are similar results to Theorem 1.5 for quadratic and antisymmetric forms, but we shall concentrate on the symmetric case in this introduction.

Theorem 1.5 has a number of important consequences. The first is that we can settle a conjecture of Berrick and Karoubi, namely we show that the map

$$\mathrm{GW}_*^s(\mathbb{Z}) \to \mathrm{GW}_*^s\left(\mathbb{Z}[\tfrac{1}{2}]\right)$$

is a 2-local equivalence, by deducing it from the fact that in K-theory and in L-theory we can control the fibre by Dévissage theorems.[3] After inverting 2, we get an equivalence

$$\mathrm{GW}_*^s(R)[\tfrac{1}{2}] \simeq \left(K_*(R)[\tfrac{1}{2}]\right)_{C_2} \oplus \mathrm{L}_{\mathrm{cl},n}^s(R)[\tfrac{1}{2}]$$

for every ring from Theorem 1.5 since the hyperbolic map is split after inverting 2. Taken together these statements combined with calculations of Berrick and Karoubi lead to the calculation of $\mathrm{GW}_*^s(\mathbb{Z})$:

Theorem 1.7. *The Grothendieck–Witt groups $\mathrm{GW}_n^s(\mathbb{Z})$ for $n > 0$ are given by*

$n =$	$\mathrm{GW}_n^s(\mathbb{Z})$
$8k$	$\mathbb{Z} \oplus \mathbb{Z}/2$
$8k+1$	$(\mathbb{Z}/2)^3$
$8k+2$	$(\mathbb{Z}/2)^2 \oplus \mathrm{K}_{8k+2}(\mathbb{Z})_{odd}$
$8k+3$	$\mathbb{Z}/w_{4k+2}$
$8k+4$	$\mathbb{Z}$
$8k+5$	0
$8k+6$	$\mathrm{K}_{8k+6}(\mathbb{Z})_{\mathrm{odd}}$
$8k+7$	$\mathbb{Z}/w_{4k+4}$

where w_{2n} is the denominator of $|\frac{B_{2n}}{4n}|$.[4]

As a last application of Theorem 1.5 we can also solve the much studied homotopy limit problem for rings of integers $\mathcal{O}_K$ in number fields (e.g. $\mathcal{O}_K = \mathbb{Z}$). Namely we show that in this case the canonical map

$$\mathrm{GW}^s(\mathcal{O}_K) \to \mathrm{K}(\mathcal{O}_K)^{hC_2}$$

is a 2-adic equivalence on connective covers. This is done by using L-theory to deduce this result from the same statement where 2 is invertible in $\mathcal{O}_K$. This latter problem has been solved by Berrick–Karoubi–Ostvaer.

[3] In fact we will prove a Dévissage theorem in GW theory

[4] The occuring K-groups $\mathrm{K}_{8k+2}(\mathbb{Z})_{odd}$ and $\mathrm{K}_{8k+6}(\mathbb{Z})_{\mathrm{odd}}$ are finite, their order can be explicitly descibed in terms of numerators of Bernoulli numbers and they are conjecturally cyclic. For $n \leq 20000$ (i.e. $k \leq 2500$) this is known by work of Weibel and it follows in all degrees if one assumes the Kummer–Vandiver conjecture.

1.2. **Poincaré-∞-categories (and overview of the lecture).** Now we want to explain a little bit more precisely how to define L-theory following a setup invented by Luire based on previous work of Ranicki. This will be covered in full details in the course and the main novel idea that leads to the new results here is to also define a version of Grothendieck–Witt theory in the same generality.

The setup is the following: we consider stable ∞-categories $\mathcal{C}$ equipped with a functor $Ϙ : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ such that the following is satisfied

(1) $Ϙ(0) = 0$, that is $Ϙ$ is reduced.
(2) There is an equivalence $D : \mathcal{C}^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{C}$ such that we have a natural equivalence
$$Ϙ(X \oplus Y) \simeq Ϙ(X) \oplus Ϙ(Y) \oplus \mathrm{map}_{\mathcal{C}}(X, DY)$$
compatible with the canonical biproduct maps.
(3) $Ϙ$ sends pushouts to totalizations. This latter condition is rather technical (and will be explained in the course) and can be ignored for now.

We shall refer to such a pair $(\mathcal{C}, Ϙ)$ as a *Poincaré-∞-category.* The reader unfamiliar with the abstract notions (which will be explained carefully in the course) should keep in mind the following example.

Example 1.8. The stable ∞-category $\mathcal{C} = \mathcal{D}^{\mathrm{perf}}(R)$ equipped with the functor
$$Ϙ^s(X) = \mathrm{map}_R(X \otimes_R X, R)^{hC_2} = \mathrm{map}_R\left((X \otimes_R X)_{hC_2}, R\right) \ .$$
i.e. the 'spectrum of bilinear forms' on X forms a Poincaré-∞-category with duality given by the usual duality of chain complexes.

Definition 1.9. *Let $(\mathcal{C}, Ϙ)$ be a Poincaré-∞-category. A* Poincaré-object *(of dimension 0) in $\mathcal{C}$ is given by an object $X \in \mathcal{C}$ together with a map $q : \mathbb{S} \to Ϙ(X)$ s.t. the induced map $\tilde{q} : X \to DX$ is an equivalence, where $\tilde{q}$ is the image of q under the map*
$$Ϙ(X) \xrightarrow{+^*} Ϙ(X \oplus X) = Ϙ(X) \oplus Ϙ(X) \oplus \mathrm{map}_{\mathcal{C}}(X, DX) \to \mathrm{map}_{\mathcal{C}}(X, DX) \ .$$
A Poincaré-object *of dimension n we have instead a map $\mathbb{S}^n \to Ϙ(X)$ such that the associated map $\tilde{q} : X[n] \to DX$ is an equivalence.*

Example 1.10. Consider the Poincaré-∞-category $(\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}), Ϙ^s)$. Then a Poincaé object of dimension n is by definition a perfect complex X over $\mathbb{Z}$ together with a symmetric bilinear form $(X \otimes_{\mathbb{Z}} X)_{hC_2} \to \mathbb{Z}[-n]$ such that the underlying form is unimodular in the sense that the map $X[n] \to DX$ is an equivalence. For example an ordinary symmetric unimodular form q on a projective module P gives rise to such a Poincaré object $(P[0], q)$ of dimension 0.

The main example to keep in mind is the following: for an n-dimensional compact, oriented manifold M we consider the cochain complex $X := C^*(M)$ and equip it with the pairing
$$(C^*(M) \otimes_{\mathbb{Z}} C^*(M))_{hC_2} \xrightarrow{\cup} C^*(M) \xrightarrow{\mathrm{ev}_{[M]}} \mathbb{Z}[-n]$$
where $[M] \in C_n(M)$ is (a representative of) the fundamental class. Thus the induced map is given by
$$\cap[M] : C^*(M)[n] \to C_*(M)$$
so that the pairing is unimodular by Poincaré duality.

Definition 1.11. *A Lagrangian (aka nullbordism) for an n-dimensional Poincaré object (X, q) in $(\mathcal{C}, Ϙ)$ is a pair consisting of a map $L \to X$ together with a path connecting $q|_L$ to 0 such that the sequence*

$$L[n] \to X[n] \cong DX \to DL$$

with the induced nullhomotopy of the composite is a fibre sequence. In this case the object (X, q) is called metabolic. The L-groups of $(\mathcal{C}, Ϙ)$ are defined as the abelian groups

$$L_n(\mathcal{C}, Ϙ) = \frac{\{\textit{Iso. classes of } n\textit{-dimensional Poincaré objects in } (\mathcal{C}, Ϙ)\}}{\{\textit{metabolic Poincaré objects}\}}$$

Remark 1.12. Note that the groups $L^s_{\mathrm{cl},n}(R)$ that we defined earlier do not quite agree with the L-groups for the Poicaré-∞-category $(\mathcal{D}^{\mathrm{perf}}(R), Ϙ^s)$. The difference is that in $L^s_{\mathrm{cl},n}(R)$ we imposed a further condition on where the chain complexes are allowed to be concentrated in degrees $[-n, 0]$) wheres for arbitrary Poincaré objects in a stable ∞-category we do not (and it would not even make sense). Apart from that the notions agree. This is a very important subtlety to which we will come back soon.

Example 1.13. Let M be a compact oriented manifold of dimension n and W be a compact oriented null-bordism of M, that is an oriented manifold of dimension $(n+1)$ together with an identification $\partial W = M$. In particular we have an inclusion $i : M \to W$ so that we get an induced map

$$i^* : C^*(W) \to C^*(M)$$

and the restriction of the form q on $C^*(M)$ to $C^*(W)$ is canonically nullhomotopic: this restriction is by naturality given by the composition

$$(C^*(W) \otimes_{\mathbb{Z}} C^*(W))_{hC_2} \xrightarrow{\cup} C^*(W) \xrightarrow{\mathrm{ev}_{i_*[M]}} \mathbb{Z}[-n]$$

But $[M] \in C_n(M)$ is the image of the canonical element $[W] \in C_{n+1}(W, M)$ under the connecting homomorphism and therefore comes with a prefered path to 0 when futher mapped to $C_*(W)$ since we have the fibre sequence $C_{*+1}(W, M) \to C_*(M) \to C_*(W)$.

We claim that this make $C^*(W) \to C^*(M)$ into a Lagrangian: the sequence in question becomes

$$C^*(W)[n] \to C^*(M)[n] \cong C_*(M) \to C_*(W) \ .$$

This being a fibre sequence is then equivalent to saying that the induced map

$$C^*(W)[n] \to \mathrm{fib}(C_*(M) \to C_*(W)) = C_*(W, M)[-1]$$

is an equivalence. This map now identifies with capping with $[W]$ so that this is indeed an equivalence by Poincaré duality for manifolds with boundary (aka. Lefschetz duality).

Together this whole discussion shows that the Poincaré objects $C^*(M)$ for a closed oriented manifold M is metabolic if M is a boundary, thus the respective element in the L-group $L_n(\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}), Ϙ^s)$ is trivial. In fact we get a graded group homomorphism

$$\Omega^{\mathrm{SO}}_* \to L_*(\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}), Ϙ^s) \qquad M \mapsto C^*(M)$$

where Ω^{SO}_* is the oriented bordism ring, i.e. Ω^{SO}_* is given by the set of all closed, oriented n-manifolds modulo the relation that ...

Now one can even construct L-theory spectra. We do not go through the construction here and only record the result, but of course we will cover that in the course.

Proposition 1.14 (Ranicki, Lurie). *For a given Poincaré-∞-category $(\mathcal{C}, \Omega)$ there exists a spectrum $L(\mathcal{C}, \Omega) \in \mathrm{Sp}$ whose homotopy groups are naturally given by the L-groups, i.e. $\pi_n L(\mathcal{C}, \Omega) \cong L_n(\mathcal{C}, \Omega)$.*

One of the main ideas that we want to employ now is to define Grothendieck–Witt groups in the same generality as L-groups. The idea is to define that similar to the definition of K_0 of a stable ∞-category where the Lagrangian sequences $L \to X \to DL$ play the role of exact sequences in the definition of $K_0(\mathcal{C})$.[5]

Definition 1.15. *Let $(\mathcal{C}, \Omega)$ be a Poincaré-∞-category. Then we define an abelian group*

$$\mathrm{GW}_0(\mathcal{C}, \Omega) = \frac{\{\textit{Iso. classes of } 0\textit{-dimensional Poincaré objects in } (\mathcal{C}, \Omega)\}}{[X] = [\mathrm{hyp}(L)] \textit{ for } L \to X \textit{ a Lagrangian}}$$

Informally the relation identitfies metabolic and hyperbolic objects.

The quotient is taken in abelian monoids and there is an implict claim here is that $GW_0(\mathcal{C}, \Omega)$ is indeed an abelian group: in fact we find that the inverse of $[X, q]$ is given by

$$[X, -q] + \mathrm{hyp}(X[1])$$

which we leave as an exercise to the reader.

Definition 1.16 (Sketch). *The higher Grothendieck–Witt groups $\mathrm{GW}_n(\mathcal{C}, \Omega)$ are the homotopy groups of the connective Grothedieck–Witt spectrum*

$$\mathrm{GW}(\mathcal{C}, \Omega) := \Omega |\operatorname{Cob}(\mathcal{C}, \Omega)|$$

where $\operatorname{Cob}(\mathcal{C}, \Omega)$ is the cobodism ∞-category of $(\mathcal{C}, \Omega)$ which is informally given as follows (and will of course be carefully defined in the course):

Objects are (-1)-dimensional Poincaré objects in $(\mathcal{C}, \Omega)$. A morphism (X, q) to (X', q') is given by a Lagrangian (aka nullbordism) of $(X \oplus X', q - q')$. One should think of the latter as a cobordism from (X, q) to (X', q') similar to the case of manifolds. This ∞-category is symmetric monoidal under direct sum so that $\Omega |\operatorname{Cob}(\mathcal{C}, \Omega)|$ really becomes a connective spectrum.

Note that the fact that the objects are (-1)-dimensional has the effect that the bordisms are (in a hopefully clear intuitive sense) 0-dimensional which is the usual convention in geometric topology: the dimension of the bordisms is what determined the "dimension" of the bordism category.

The pleasant feature of this setup is that one has an abstract version of the fibre sequence of Theorem 1.5.

Theorem 1.17 (#nine). *For every Poincaré-∞-category we have a fibre seuqence*

$$\mathrm{K}(\mathcal{C})_{hC_2} \to \mathrm{GW}(\mathcal{C}, \Omega) \to \tau_{\geq 0} L(\mathcal{C}, \Omega)$$

of connective spectra. Moreover the functors GW and L both satisfy universal properties similar to the ones given for K-theory of stable ∞-categories by Bliumberg–Gepner–Tabuada: informally speaking GW is the universal additive functor and L the universal bordism invariant functor (details in the course).

[5] Recall that $K_0(\mathcal{C})$ for a stable ∞-category is given by the monoid of isomorphism classes in $\mathcal{C}$ modulo the relation that for an exact sequence $A \to B \to C$ we have $[B] = [A] + [\oplus[C]$.

Remark 1.18. We want to remark that the actual definition of $\mathrm{GW}(\mathcal{C}, \Qoppa)$ that we will cover in the course leads to a non-connective spectrum (whose connective cover is the GW what we have defined here) and with this non-connective spectrum the abstract theorem 1.17 holds without taking the connective cover on L-theory (the left hand term is still connective K-theory). In particular this implies that the negative homotopy groups of this non-connective spectrum are isomorphic to the L-groups.

One should consider this Theorem as a blueprint for the fibre sequence we actually care about (Theorem 1.5). Now finally the question is how one can deduce Theorem 1.5 above from the abstract Theorem 1.17. To this end one has to choose the correct Poincaré-∞-category $(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^g)$. Here $\Qoppa^g$ is a functor

$$\Qoppa^g : \mathcal{D}^{\mathrm{perf}}(R)^{\mathrm{op}} \to \mathrm{Sp}$$

which we call the *genuine* quadratic functor. It is obtained as the non-abelian derived functor (in the sense of Dold–Puppe) of the functor of symmetric bilinear forms on a projective module:

$$\mathrm{Proj}_R^{\mathrm{op}} \to \mathrm{Ab} \qquad P \mapsto \mathrm{Hom}_R(P \otimes_R P, R)^{C_2} \ .$$

We will explain this process in detail later. It is a procedure that can be applied to non-additive functors similar to deriving in the usual context. The key is to note that the functor $\Qoppa^g$ is completely different from the functor $\Qoppa^s$ given by 'homotopical bilinear forms'. In particular $\Qoppa^g$ has much better connectivity properties which make it possible to perform an algebraic variant of surgery to Poincaré objects for $(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^g)$ resulting in the following:

Theorem 1.19. *For every commutative ring R there are natural equivalences of connective spectra*

$$\mathrm{GW}(R) \simeq \mathrm{GW}(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^g) \qquad \textit{(Hebestreit–Steimle)}$$
$$\mathrm{L}^s_{\mathrm{cl}}(R) \simeq \tau_{\geq 0} L(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^g) \qquad \textit{(\#nine)}$$

Then combining Theorems 1.17 and 1.19 immediately leads to Theorem 1.5. We also note that the first equivalence is, if we replace our definition of GW by its proper non-connective version, also of the form $\mathrm{GW}(R) \simeq \tau_{\geq 0} \mathrm{GW}(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^g)$.

Remark 1.20. Let us mention again that everything said so far, with the exception of the solution of the homotopy limit problem, even works for quadratic and ϵ-symmetric forms instead of symmetric forms. This also nicely fits into the framework of Poincaré-∞-categories and will be covered in the course as well. In particular we will prove the surpising fact that the map $\mathrm{GW}^q(\mathbb{Z}) \to \mathrm{GW}^s(\mathbb{Z})$ is an isomorphism in degrees ≥ 2 where the source is the Grothendieck–Witt spectrum based on quadratic instead of symmetric forms.

2. Preliminaries on stable ∞-categories

As announced, we will assume that everyone is familiar with the notion of an ∞-category and the notion of limits and colimits therein. We will assume knowledge of some examples such as the ∞-category of spaces $\mathcal{S}$ and the ∞-category of spectra Sp. For an ∞-category $\mathcal{C}$ and objects X, Y we denote the mapping space by

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathcal{S} \ .$$

The homotopy category of $\mathcal{C}$ will be denoted by $\mathrm{Ho}(\mathcal{C})$ and we have that homotopy classes of morphisms $X \to Y$ are given by

$$\pi_0(\mathrm{Map}_{\mathcal{C}}(X,Y)) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y)\ .$$

We will occasionally also write this as $[X,Y]$. We write $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ for the ∞-category of functors between ∞-categories $\mathcal{C}$ to $\mathcal{D}$.

We will also sometimes use construction principles such as the homotopy coherent nerve (which produces an ∞-category $N_\Delta\mathcal{C}$ from a simplicially enriched category $\mathcal{C}$). For example $\mathcal{S}$ is the nerve of the simplicially enriched category of Kan complexes. We shall also use the Dwyer-Kan localization of an ∞-category $\mathcal{C}$ at a class of weak equivalences in $\mathcal{C}$ (i.e. a set of 1-morphisms) which we denote by

$$\mathcal{C} \to \mathcal{C}[W^{-1}]$$

and which has is defined as the universal ∞-category in which the morphisms in W map to equivalences.

Definition 2.1. (1) *An ∞-category $\mathcal{C}$ is pointed, if it admits an object $0 \in \mathcal{C}$ that is initial and terminal at the same time. In this case we have for every pair of objects X, Y a canonical morphism $0 : X \to Y$ defined as the factorization through 0.*

(2) *A pointed ∞-category $\mathcal{C}$ is* semiadditive *if it admits finite products and finite coproducts and for every pair of objects X, Y the canonical map*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : X \amalg Y \to X \times Y$$

is an equivalence. In this case we write $X \oplus Y$ for the biproduct.

(3) *A semiadditive ∞-category $\mathcal{C}$ is additive if moreover for every X the shearing map*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : X \oplus X \to X \oplus X$$

is an equivalence.

(4) *A pointed ∞-category $\mathcal{C}$ is stable if it admits finite limits and colimits and a square in $\mathcal{C}$ is a pushout precisely if it is a pullback. Recall that a square is a functor $\Delta^1 \times \Delta^1 \to \mathcal{C}$, i.e comes with a filling homotopy. In this case we refer to such squares as exact.*

Example 2.2. The ∞-category of pointed spaces $\mathcal{S}_*$ is pointed with zero object the pointed space pt. The (nerve) of the ordinary category of abelian groups Ab is additive. The ordinary category Proj_R of finite projective modules over a ring R is additive. The ∞-category of spectra is stable.

Definition 2.3. *In every stable ∞-category $\Sigma : \mathcal{C} \to \mathcal{C}$ is invertible with inverse Ω since the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

is a pushout, thus a pullback so that $\Omega\Sigma X \simeq X$. For $n \in \mathbb{Z}$ and $X \in \mathcal{C}$ we will write $X[n]$ for the n-fold application of Σ to X. Fibre sequences in $\mathcal{C}$ are the same as cofibre sequences and will simply be refered to as exact sequences.

We note that for an additive ∞-category $\mathcal{C}$ the homotopy category is also additive. It then follows that it is canonically enriched over abelian groups, i.e. that $[X, Y]$ carries the structure of an abelian groups and composition is bilinear.

Lemma 2.4. *Every stable ∞-category is additive.*

Proof. A pointed ∞-category with all finite colimits and limits is preadditive precisely if the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "(\mathrm{id},0)"] \\ Y \arrow[r, "(0,\mathrm{id})"'] & X \times Y \end{tikzcd}$$

is a pushout. If $\mathcal{C}$ is stable the for preadditivity it is enough to show that it is a pullback. To see this we consider the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y \arrow[r] & X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & 0 \end{tikzcd} .$$

The lower right square is a pullback and the right outer one, thus also the outer right one. Since also the outer horizontal square is a pullback it follows that the left hand square is one.

To see that $\mathcal{C}$ is additive we have to show that also the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "(\mathrm{id},0)"] \\ X \arrow[r, "(\mathrm{id},\mathrm{id})=\Delta"'] & X \times X \end{tikzcd}$$

is a pushout, hence a pullback. This follows by the exact same argument. □

It follows that homotopy classes of maps in a stable ∞-category carry the structure of an abelian group. We soon will see that in fact the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ canonically refine to mapping spectra $\mathrm{map}_{\mathcal{C}}(X, Y)$ for a stable ∞-category, so that the abelian group structure on its π_0 comes from this.

Definition 2.5. *A functor between pointed ∞-categories is called reduced, it it preserves the basepoint (note that is a property). A reduced functor between stable ∞-categories is called exact if it sends exact squares to exact squares.*

Lemma 2.6. *For a functor $\mathcal{C} \to \mathcal{D}$ between stable ∞-categories the following are equivalent*

(1) *It is exact.*
(2) *It preserves the zero object and pushouts.*
(3) *It preserves the zero object and pullbacks.*
(4) *It preserves finite colimits*
(5) *It preserves finite limits.*

Proof. Clearly (1) ⇔ (2) ⇔ (3). Now since every finite colimit is an iterated pushout we get that (2) ⇔ (4) and dually for (3) ⇔ (5). □

Only give a sketch and not an exhaustive treatment.

Proposition 2.7. *Let R be associative ring* [6] *. Then we have an ∞-category*

$$\mathcal{D}(R) = \mathrm{N}\,\mathrm{Ch}(R)[q\text{-}iso^{-1}]$$

obtained from the ordinary category of chain complexes by formally inverting the quasi-isomorphisms between chain complexes. This ∞-catergory is stable and the shift $[n]$ is given by shifting the chain complex by n.

Proof. First of all, we consider an auxiliar ∞-category

$$\mathcal{K}(R) := \mathrm{N}\,\mathrm{Ch}(R)[\text{ch-eq}^{-1}]\ .$$

The point is that this ∞-category also admits a very concrete description as follows: consider $\mathrm{Ch}(R)$ as a category enriched over chain complexes using the Hom-complex. Then apply Dold-Kan to the connective cover of these chain complexes to turn it into a simplicially enriched category and apply the homotopy coherent nerve. This also produces $\mathcal{K}(R)$ which follows from the existence of cylinder objects and Proposition 1.3.4.7. in Higher Algebra. From this second description it is (more or less) easy to give explict formulas for pullbacks and pushouts using mapping cones. This then implies that $\mathcal{K}(R)$ is stable and the suspension is given by the shift.

Now it is clear that $\mathcal{D}(R)$ can be obtained from $\mathcal{K}(R)$ by further localizing at the quasi-isomorphisms since we have that

$$\mathcal{K}(R))[\text{q-iso}^{-1}] = \left(\mathrm{N}\,\mathrm{Ch}(R)[\text{ch-eq}^{-1}]\right)[\text{q-iso}^{-1}] = \mathrm{N}\,\mathrm{Ch}(R)[\text{q-iso}^{-1}] = \mathcal{D}(R)$$

where the middle equivalence follows by comparing universal properties (using that every chain equivalence is a quasi-iso).

Now the point is that in $\mathcal{K}(R)$ the class if quasi-isomorphism has very nice properties: it contains all equivalences and is closed under pushouts and pullbacks. Now we claim that in general if we have a stable ∞-category $\mathcal{C}$ with such a class of weak equivalences W then $\mathcal{C}[W^{-1}]$ is also stable, the functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ is exact and the mapping space in $\mathcal{C}[W^{-1}]$ is given by the filtered colimit

$$\mathrm{Map}_{\mathcal{C}[W^{-1}]}(X,Y) \simeq \varinjlim_{\tilde{X} \xrightarrow{\simeq} X} \mathrm{Map}_{\mathcal{C}}(\tilde{X}, Y)\ .$$

This fact is proven in Nikolaus-Scholze Theorem I.3.3. .

Sidenote: using the formula for the mapping space in the DK-localization, it is not a priori clear, that this is a small space, i.e. that $\mathcal{C}[W^{-1}]$ is locally small, even if $\mathcal{C}$ is locally small. In our case this is true however since the ∞-category $\tilde{X} \xrightarrow{\simeq} X$ admits a cofinal, small subcategory. In fact one can show that the functor $\mathcal{K}(R) \to \mathcal{D}(R)$ admits a left adjoint by $\mathcal{K}$-projective replacement.[7] This right adjoint is then fully faithful and shows that $\mathcal{D}(R)$ is a full subcategory of $\mathcal{K}(R)$. □

The last assertion not only works for the mapping space in $\mathcal{D}(R)$ but also gives formulas for derived functors: if $F : \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{D}$ is a functor then the right derived functor

$$RF : \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}$$

[6] From now on our rings will always be possibly non-commutative in contrast to the first section

[7] The $\mathcal{K}$-projectives are precisely the colocal objects, i.e. this $P \in \mathcal{K}(R)$ such that $\mathrm{Map}_{\mathcal{K}}(P, -)$ preserves quasi-isomorphisms.

defined as the left Kan extension along $\mathcal{K}(R)^{\mathrm{op}} \to \mathcal{D}(R)^{\mathrm{op}}$ is given by $LF(X) = \underrightarrow{\mathrm{colim}}_{\tilde{X} \xrightarrow{\simeq} X} F(\tilde{X})$ provided $\mathcal{D}$ has all necessary colimits. We e.g. get the derived Hom-complex $R\,\mathrm{Hom}$. Dually the left derived functor LF for a functor $F : \mathcal{K}(R) \to \mathcal{D}$ defined as the right Kan extension can be obtained as

$$LF : \mathcal{D}(R) \to \mathcal{D}$$

with the formula $LF(X) = \underleftarrow{\lim}_{\tilde{X} \xrightarrow{\simeq} X} F(\tilde{X})$. This for example gives the derived tensor product.

Remark 2.8. The full subcategory $\mathcal{K}(R)^{\mathrm{q-acyc}} \subseteq \mathcal{K}(R)$ consisting of objects that are quasi-isomorphic to 0 is a stable subcategory in the sense that it is closed under finite limits and colimits. A map $f : X \to Y$ is a quasi-iso precisely if its fibre lies in this subcategory. Thus $\mathcal{D}(R)$ is the Verdier Quotient of $\mathcal{K}(R)$ by $\mathcal{K}(R)^{\mathrm{q-acyc}}$. Verdier Quotients will play a very important role later in the lecture.

Another way of thinking about $\mathcal{D}(R)$ is as the ∞-category of module spectra over the Eilenberg-MacLane spectrum $HR \in \mathrm{Sp}$ but we will not really detail that here.

When we speak about chain complexes we shall always consider them as objects in $\mathcal{D}(R)$. In particular by an equivalence we shall always mean an equivalence in $\mathcal{D}(R)$, which corresponds by what we have shown above to zig-zags $X \leftarrow X' \to Y$ of quasi-isomorphisms in $\mathrm{Ch}(R)$.

Lemma 2.9. *For a chain complex $X \in \mathcal{D}(R)$ the following are equivalent:*

(1) *It can be represented up to equivalence by a finite length chain complex of finite projective R-modules.*
(2) *It lies in the smallest subcategory $\mathcal{D}^{\mathrm{perf}}(R) \subseteq \mathcal{D}(R)$ that contains $R[0]$, is stable and closed under retracts.*
(3) *It is dualizable in the sense that the canonical map*

$$\mathrm{RHom}_R(X, R) \otimes^L_R Y \to \mathrm{RHom}_R(X, Y) \qquad f \otimes x \mapsto f \cdot x$$

is an equivalence for every $Y \in \mathcal{D}(R)$. Here $\mathrm{RHom}_R(X, R)$ is considered as a chain complex of right R-modules.
(4) *It is a compact object in $\mathcal{D}(R)$, that is $\mathrm{Map}_{\mathcal{D}(R)}(X, -) : \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{S}$ commutes with filtered colimits*

Definition 2.10. *We call chain complexes satisfying one of these equivalent conditions perfect. We denote the full subcategory of those by $\mathcal{D}^{\mathrm{perf}}(R) \subseteq \mathcal{D}(R)$.*

Proof of Lemma 2.9. (1) $\Rightarrow$ (2) We note that $\mathcal{D}^{\mathrm{perf}}(R)$ is by definition closed under sums and retracts, thus it in particular contains $P[0]$ for P a projective R-module (here we have used that direct sums are formed underlying). Since it is stable, it is also closed under shifts, thus contains $P[n]$. Now a chain complex of length 2

$$X = (... \to 0 \to P \to Q \to 0 \to ...)$$

with P in degree 0 and P, Q finitely generated projective sits in a fibre sequence

$$Q[0] \to X \to P[1]$$

and thus lies in $\mathcal{D}^{\mathrm{perf}}(R)$. A similar inductive argument then shows the claim.
(2) $\Rightarrow$ (3) The class of all X for which

$$\mathrm{RHom}_R(X, R) \otimes^L_R Y \to \mathrm{RHom}_R(X, Y) \qquad f \otimes x \mapsto f \cdot x$$

is an equivalence for all Y is a stable subcategory closed under retracts. It clearly contains R and thus also X.
(3) $\Rightarrow$ (4) The functor Map is the underlying space of $\mathrm{RHom}_R(X,Y)$. Thus it suffices to show that $\mathrm{RHom}_R(X,-)$ commutes with filtered colimits. But this is equivalent to $\mathrm{RHom}_R(X,R) \otimes_R^L -$ which clearly does.
(4) $\Rightarrow$ (1) We claim that every object in $\mathcal{D}(R)$ can be written as a filtered colimit of finite length chain complex of finite projective R-modules. This is left as an exercise for the reader. Thus we have a description $X = \varinjlim X_i$. The identity map $X \to X$ now has to factor by compactness through one of the X_i's so that we find that X is a retract of this X_i. Since we are additive we get that

$$X_i = X \oplus Y$$

for some chain complex Y. Now the claim follows from the assertion that if a binary sum of chain complexes can be represented by a bounded finite projective complex, then so can the summands (Lemma 15.70.5 in the Stacks project). □

Remark 2.11. The way the last assertion is shown in the stacks project is to use another equivalent description of perfect complexes: these are precisely the complexes that are pseudo-coherent (i.e. bounded below and of finite type in topology terms) and of bounded Tor-Amplitude. Here a complex $X \in \mathcal{D}(R)$ is said to have Tor-Amplitude in $[a,b]$ for $a,b \in \mathbb{Z}$ if for any right R-module M we have

$$H_i(M[0] \otimes_R^L X) = \mathrm{Tor}_i^R(M[0], X) = 0\ .$$

for $i \notin [a,b]$. Then one also finds the following important description:

A complex $X \in \mathcal{D}(R)$ is perfect with Tor-Amplitude in $[a,b]$ precisely if it can be represented by a finite projective chain complex

$$... \to 0 \to P_b \to P_{b-1} \to \ldots \to P_a \to 0 \to ...$$

supported in the interval $[a,b]$ (see Lemma 15.70.2 in Stacks project).

Remark 2.12. There is another stable subcategory of $\mathcal{D}(R)$ that will a play a role here, namely the one generated by $R \in \mathcal{D}(R)$ without allowing retracts. This will be denoted by $\mathcal{D}^{\mathrm{fp}}(R) \subseteq \mathcal{D}(R)$ and objects will be called finitely presented. Being finitely presented for a chain complex X is equivalent to being representable by a finite lenght chain complex of finite free modules. However this is class does not admit a description intrinsic to the ∞-category $\mathcal{D}(R)$ like perfect complexes.

Remark 2.13. One can also study similar notions of perfectness in spectra (or for modules over an arbitrary ring spectrum). For example the compact objects in Sp, denoted Sp^ω are also characterized as the smallest stable subcategory $\mathrm{Sp}^{\mathrm{fp}} \subseteq \mathrm{Sp}$ containing the sphere (which is then automatically closed under retracts as the respective Wall finiteness obstruction vanishes). We will simply write this ∞-category as $\mathrm{Sp}^{\mathrm{fin}}$. It will play an important role in this course.

Corollary 2.14. *For any two strictly perfect complexes X, Y (that is finite length, finite projective) the mapping space $\mathrm{Map}_{\mathcal{D}(R)}(X,Y)$ is the space underlying the Hom-complex $\underline{\mathrm{Hom}}_R(X,Y)$. Also we have that*

$$\mathcal{D}^{\mathrm{perf}}(R) \simeq \mathrm{Ch}^{\mathrm{st-perf}}[q.iso^{-1}]$$

For any ring R the functor

$$D : \mathcal{D}^{\mathrm{perf}}(R)^{\mathrm{op}} \to \mathcal{D}^{\mathrm{perf}}(R^{\mathrm{op}}) \qquad X \mapsto \mathrm{RHom}_R(X,R)$$

is an equivalence of ∞*-categories.* [8]

Proof. As shown in the proof of Propositon 2.9 we see that $\mathcal{D}(R)$ embedds fully faithfully into $\mathcal{K}(R)$ as the $\mathcal{K}$-projective objects. Since every finite length finite projective complex is $\mathcal{K}$-projective it follows that the mapping complex in $\mathcal{D}(R)$ can be computed as the underlying space of the Hom-complex which shows the first claim.

For the second assertion we note that the mapping spaces in the ∞-category $\mathrm{Ch}^{\mathrm{perf}}[\mathrm{q.iso}^{-1}]$ can be described as the space underlying the mapping chain complex between two projective complexes. By the first part this is also the mapping space in $\mathcal{D}^{\mathrm{perf}}(R)$.

The third claim immediately follows since the one-categorical functor

$$\mathrm{Ch}^{\mathrm{st-perf}}(R)^{\mathrm{op}} \to \mathrm{Ch}^{\mathrm{st-perf}}(R^{\mathrm{op}})$$

is an equivalence and preserves and reflect quasi-isos. □

Lemma 2.15. *For any stable* ∞*-category* $\mathcal{C}$ *the the functor*

$$\Omega^\infty : \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathcal{S})$$

is an equivalence of ∞*-categories. Here the first* ∞*-category denotes the* ∞*-category of functors that are exact and the second denotes the* ∞*-category of functors that are finite limit preserving (i.e. left exact).*

Proof. Let us give a way to construct the inverse assigning to a functor $F : \mathcal{C} \to \mathcal{S}$ the functor $F' : \mathcal{C} \to \mathrm{Sp}$. Since the functor F' has to be exact we have to have that

$$\Omega^{\infty-n} F'(X) = \Omega^\infty F(X)[n] = \Omega^\infty F(X[n])$$

Thus n-th space of the spectrum $F'(X)$ has to be given by the space

$$\Omega^\infty F(X[n])$$

with basepoint given by the point $\mathrm{pt} = F(0) \to F(X[n])$. The structure maps are then the canonical equivalences

$$\Omega F(X[n+1]) \simeq F(\Omega F(X[n+1]) = F(X[n]) \ .$$

Now all of this is natural and gives the proof. Note that we have some details in the last step omitted. □

Proposition 2.16. *For every stable* ∞*-category* $\mathcal{C}$ *we have a unique mapping spectrums functor*

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp} \qquad (X, Y) \mapsto \mathrm{map}_{\mathcal{C}}(X, Y)$$

which is exact in both variables separately and which refines the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ *in the sense that we have a natural (in* X *and* Y*) equivalence*

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \Omega^\infty \mathrm{map}_{\mathcal{C}}(X, Y) \ .$$

Proof. We first claim that the functor

$$\Omega^\infty : \mathrm{Fun}^{\mathrm{bi-ex}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{bi-Lex}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})$$

[8]Here R^{op} denotes the ring obtained from R with opposite multiplication, i.e. $(r, s) \mapsto sr$. Left modules over R^{op} are the same as right R-modules, so that $\mathcal{D}^{\mathrm{perf}}(R^{\mathrm{op}})$ can equivalently be described by chain complexes of right R-modules.

is an equivalence for $\mathcal{C}$ stable. Here the first ∞-category denotes the ∞-category of functors that are exact in each variable separately. The second denotes the ∞-category of functors that are finite limit preserving (i.e. left exact) in each variable separately. This follows from the previous Lemma using that

$$\mathrm{Fun}^{\mathrm{bi-ex}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Sp}))$$

and

$$\mathrm{Fun}^{\mathrm{bi-Lex}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{Lex}}\left(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathcal{S})\right)$$

Then $\mathrm{Map}_{\mathcal{C}}$ is an element in the latter and so uniquely lifts to the former. □

Example 2.17. There is a functor

$$H : \mathcal{D}(R) \to \mathrm{Sp}$$

which assigns to a chain complex its 'underlying' spectrum. It is corepresented by R, i.e. of the form $\mathrm{map}_{\mathcal{D}(R)}(R, -)$. When applied to R it is given by the Eilenberg-MacLane spectrum HR. Since limits of spectra can be detected on the underlying spaces it follows that this functor is limit preserving.

Example 2.18. For any object $M \in \mathcal{D}(R)$ we have a functor $\mathcal{D}(R)^{\mathrm{op}} \to \mathrm{Sp}$ given by $\mathrm{map}_{\mathcal{D}(R)}(-, M)$. This functor is also limit preserving, i.e. sends colimits in $\mathcal{D}(R)$ to limits of spectra and will be denoted by $\underline{M}$ since it is a form of Yoneda embedding for stable ∞-categories. In fact, every limit preserving functor $\mathcal{D}(R)^{\mathrm{op}} \to \mathrm{Sp}$ is of this form since such limit preserving functors are the same as limit preserving functors $\mathcal{D}(R)^{\mathrm{op}} \to \mathcal{S}$ and the latter are by [**?**] representable. In fact we get an equivalence between $\mathcal{D}(R)$ and limit preserving functors. Upon restricting to $\mathcal{D}^{\mathrm{perf}}(R)$ this given an equivalence

$$\mathcal{D}(R) \simeq \mathrm{Fun}^{\mathrm{lim}}(\mathcal{D}(R)^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{\mathrm{perf}}(R)^{\mathrm{op}}, \mathrm{Sp})$$

which will be important later.

3. Quadratic functors

Now we want to discuss a class of functors called quadratic which is a special case of Goodwillie calculus.

If we think of a stable ∞-category $\mathcal{C}$ as an analogue of a vector space. Then exact functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ are linear functions on $\mathcal{C}$. We will also refer to exact functors as linear and denote those by $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C})$. Quadratic functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ correspond in this analogy to reduced polynomials of degree 2. In this section $\mathcal{C}$ will be stable throughout.

Definition 3.1. *For a given reduced functor $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ we define an associated functor*

$$B_{\Omega} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$$

called the (second) cross effect of Ω by letting $B_{\Omega}(X, Y)$ be the complement of the split inclusion

$$\Omega(X) \oplus \Omega(Y) \to \Omega(X \oplus Y) \to \Omega(X) \oplus \Omega(Y)\ .$$

In fact we see that we immediately get equivalences

$$B_{\Omega}(X, Y) \simeq B_{\Omega}(Y, X)\ .$$

It turns out that these equivalences are coherent in terms of the following definition.

Definition 3.2. *A symmetric functor* $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ *is a homotopy fixed point for the action of* C_2 *on* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ *or equivalently a functor* $(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}})_{hC_2} \to \mathrm{Sp}$. *A functor* $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ *is called bilinear, if it is exact in each variable separately. We denote symmetric bilinear functors by*

$$\mathrm{Fun}^{\mathrm{s}}(\mathcal{C}) := \mathrm{Fun}^{\mathrm{Bi-exc}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{hC_2} \ .$$

Lemma 3.3. *For every* $\Omega \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ *the functor* $B_\Omega : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ *admits a canonical refinement to a symmetric functor and the canonical maps*

$$\begin{tikzcd} & \Omega(X) \arrow[d, "\Delta_*"] & \\ B_\Omega(X,X) \arrow[r] & \Omega(X \oplus X) \arrow[r] \arrow[d, "\nabla_*"] & B_\Omega(X,X) \\ & \Omega(X) & \end{tikzcd}$$

are C_2*-equivariant for every* X *(here* C_2 *acts trivially on* $\Omega(X)$ *and in the 'obvious' way on the other terms).*

Proof. The retract diagram

$$\Omega(X) \oplus \Omega(Y) \to \Omega(X \oplus Y) \to \Omega(X) \oplus \Omega(Y) \ .$$

is a diagram in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{hC_2}$ as we let X and Y vary, thus so is B_Ω and the maps $B_\Omega(X,Y) \to \Omega(X \oplus Y) \to B_\Omega(X,Y)$ are equivariant. Upon restriction along the diagonal $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}$ this implies the horizontal sequence. The vertical one is clear □

We thus obtain canonical maps

$$B_\Omega(X,X)_{hC_2} \to \Omega(X \oplus X)_{hC_2} \to \Omega(X) \to \Omega(X \oplus X)^{hC_2} \to B_\Omega(X,X)^{hC_2}$$

whose composition is the norm of the C_2-object $B_\Omega(X,X)$. [9]

Definition 3.4. *For a given reduced functor* $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ *we consider the cofibre*

$$B_\Omega(X,X)_{hC_2} \to \Omega(X) \to L_\Omega(X)$$

which is a functor $L_\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$. *We call the functor* Ω quadratic *if* L_Ω *is linear and* B_Ω *is bilinear. In this case we refer to* L_Ω *as the linear part of* Ω. *We denote the full subcategory of quadratic functors by*

$$\mathrm{Fun}^q(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \ .$$

A quadratic functor Ω *is called* homogenous *if* $L_\Omega = 0$.

Remark 3.5. A functor Ω is quadratic, precisely if it is reduced and 2-excisive in the sense of Goodwillie, i.e. sends strongly coCartesian 2-cubes to Cartesian 2-cubes. This easily follows from standard facts about Goodwillie calculus and cross effects.

Example 3.6. Any exact functor $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is quadratic. To see this note that in this case $B_\Omega = 0$ (which is in particular bilinear) and thus $L_\Omega = L$, which is linear.

[9] Recall that for every C_2-spectrum Z the norm is the map $Z_{hC_2} \to Z^{hC_2}$ which informally sends z to $z + \sigma z$.

Example 3.7. Let B be a symmetric bilinear functor $B : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$. Then we consider the functors

$$\Qoppa^q_B(X) = B(X,X)_{hC_2} \qquad \text{and} \qquad \Qoppa^s_B(X) = B(X,X)^{hC_2}$$

The q here is for quadratic and the s for symmetric (we will understand this notation later). We claim that both of these functors are quadraic. In the first case we get that

$$\begin{aligned}\Qoppa^q_B(X \oplus Y) &= B(X\oplus Y, X\oplus Y)_{hC_2} \\ &= B(X,X)_{hC_2} \oplus \left(B(X,Y)\oplus B(Y,X)\right)_{hC_2} \oplus B(Y,Y)_{hC_2} \\ d &= \Qoppa^q_B(X) \oplus B(X,Y) \oplus \Qoppa^q_B(Y)\end{aligned}$$

so that $B_{\Qoppa^q_B} = B$. It follows that $L_{\Qoppa^q_B}$ which then shows that $\Qoppa^q_B$ is quadratic. For the symmetric functor we also find with the same argument that $B_{\Qoppa^s_B} = B$ and thus that

$$L_{\Qoppa^s_B}(X) = \Qoppa^s_B(X)/\Qoppa^q_B(X) = B(X,X)^{tC_2}.$$

To fact that $\Qoppa^s_B$ is quadratic then follows from the fact that the functor $B(X,X)^{tC_2}$ is exact in X. This is a standard argument which we will give below after the examples, see Corollary 3.13.

Example 3.8. Now we consider a subexample of the last example, which will be the most important example for this lecture. Let us first assume that R is commutative. Then we consider the symmetric bilinear functor on $\mathcal{D}^{\mathrm{perf}}(R)$ given by

$$B(X,Y) = \mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R)}(X \otimes^L_R Y, R)$$

In this case the last example leads to the functors

$$\Qoppa^q_R := \Qoppa^q_B \qquad \text{and} \qquad \Qoppa^s_R := \Qoppa^s_B\ .$$

Definition 3.9. *A ring with involution is given by a ring together with an isomorphism $\sigma : R \to R^{\mathrm{op}}$ such that $\sigma^2 = \mathrm{id}$.*

Example 3.10. Every commutative ring R is canonically a ring with involution where $\sigma = \mathrm{id}$. We want to generalize Example 3.8 to the case of rings with involutions. The involution can be used to turn right modules (aka R^{op}-modules) into left R-modules. In particular the $R-R$-bimodule R can be consider as a $R\otimes R$-module. Explicitly the left $R\otimes R$-action is given by

$$(r\otimes s)\cdot t \mapsto rt\sigma(s).$$

Moreover this module is fixed under the flip action, i.e. the restriction along $\tau : R\otimes R \to R\otimes R$ is isomorphic to itself in a symmetric way, i.e. a homotopy fixed point for the flip action on $\mathcal{D}(R\otimes R)$. Now we consider the symmetric biliinear functors

$$B(X,Y) = \mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R\otimes^L_{\mathbb{Z}} R)}(X\otimes_{\mathbb{Z}} Y, R) \in \mathrm{Sp}\ .$$

then we also get quadratic functors

$$\Qoppa^q_R, \Qoppa^s_R : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}\ .$$

Now we want to give the argument for the fact that for any symmetric bilinear functor $B \in \mathrm{Fun}^s(\mathcal{C})$ the functor

$$B(X,X)^{tC_2}$$

is exact. We first observe that this functor is obviously additive in since

$$\begin{aligned} B(X\oplus Y, X\oplus Y)^{tC_2} &= B(X,X)^{tC_2} \oplus (B(X,Y)\oplus B(Y,X)) \oplus B(Y,Y)^{tC_2} \\ &= B(X,X)^{tC_2} \oplus B(Y,Y)^{tC_2} . \end{aligned}$$

Now we want to establish a criterion to deduce that an additive functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is exact to show that $B(X,X)^{tC_2}$ is indeed additive. This will be useful for later (even though it might be a bit overkill here). We will first need some terminology: recall that a simplicial object $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ is called n-skeletal or n-dimensional if it is equivalent to the left Kan extension of its restriction to $\Delta^{\mathrm{op}}_{\leq n} \subseteq \Delta^{\mathrm{op}}$. In this case the colimit $\mathrm{colim}_{\Delta^{\mathrm{op}}} X_\bullet$ is equivalent to the colimit

$$\mathrm{colim}_{\Delta^{\mathrm{op}}_{\leq n}} \left(X_\bullet|_{\Delta^{\mathrm{op}}_{\leq n}} \right)$$

of the restriction of $X_\bullet$ to $\Delta^{\mathrm{op}}_{\leq n}$. For $n=1$ the colimit over $\Delta^{\mathrm{op}}_{\leq 1}$ is called a reflexive coequalizer and can thus we computed as the colimit over the associated 1-dimensional simplicial object. Every pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

gives rise to a reflexive coqualizer $B \oplus A \oplus C \rightrightarrows B \oplus C$.

Definition 3.11. *Let $\mathcal{C}$ be a stable ∞-category. We say that a functor* preserves finite totalizations *if for any finite dimensional simplicial object $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ the canonical map*

$$F(\mathrm{colim}_{\Delta^{\mathrm{op}}} X_\bullet) \to \lim_{\Delta} F(X_\bullet)$$

is an equivalence. We say it sends pushouts to totalizations *if this is true for any 1-dimensional simplicial object induced by a pushout square.*

In other words: it sends a coequalizer diagram

$$X_1 \rightrightarrows X_0 \to X_{-1}$$

to a totalization

$$F(X_{-1}) \to F(X_0) \rightrightarrows F(X_1) \to F(X_1 \oplus_{X_0} X_1) \to \dots .$$

Lemma 3.12. *For an additive functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ TFAE*

(1) *It is exact*
(2) *It preserves finite totalizations*
(3) *It sends pushouts to totalizations.*

Proof. Assume F is exact. Then the diagram $F(X_\bullet)$ for $X_\bullet$ n-dimensional is equivalent to the right Kan extension of its restriction to $\Delta_{\leq n}$. This follows from the fact that F preserves finite limits and that these Kan extensions are pointwise given by finite limits (as one can see from the pointwise formulas). Thus the limit of $F(X_\bullet)$ is the same as the limit of its restriction to $\Delta_{\leq n}$. Thus the claim that F preserves finite totalizations follows from the fact that $\Delta_{\leq n}$ is finite and F preserves finite limits

Conversely assume that F is additive and preserves finite totalizations. We consider a pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{C}$. We an write $D = B \oplus_A C$ as the geometric realization of the 1-dimensional simplicial object

$$\ldots \to B \oplus A \oplus A \oplus C \longrightarrow B \oplus A \oplus C \longrightarrow B \oplus C\ .$$

It follows that $F(D)$ is the totalization of the simplicial object

$$F(B) \oplus F(C) \to F(B) \oplus F(A) \oplus F(C) \to ...$$

i.e. the pullback $F(B) \times_{F(A)} F(C)$ which shows exactness. □

Corollary 3.13. *For every symmetric bilinear functor $B \in \mathrm{Fun}^s(\mathcal{C})$ the associated functor*

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Sp} \qquad X \mapsto B(X, X)^{tC_2}$$

is exact.

Proof. For a 1-dimensional simplicial object $X_\bullet \in \mathcal{C}$ we see that the diagonal cosimplicial object

$$n \mapsto B(X_n, X_n)$$

is 2-dimensional (considered as a simplicial object in $(\mathrm{Sp}^{BC_2})^{\mathrm{op}}$) with the same argument that shows that the product of two 1-dimensional simplicial sets is 2-dimensional. Moreover its limit is given by

$$\lim_n B(X_n, X_n) = \lim_{i,j} B(X_i, X_j) = B(\operatorname{colim} X_i, \operatorname{colim} Y_i)$$

Thus applying any exact functor to it commutes with this totalization. Applying this to $(-)^{tC-2}$ shows the claim. □

Proposition 3.14. *Let $\text{♀} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ be a reduced functor for which $B_{\text{♀}}$ is biadditive.* [10] *Then TFAE*

(1) *It is quadratic*
(2) *It preserves finite totalizations*
(3) *It sends pushouts to totalizations.*

Proof. For (1)⇒(2) observe that if ♀ is quadratic, then it sits in the exact sequence

$$B(X, X)_{hC_2} \to \text{♀}(X) \to \mathrm{L}_{\text{♀}}(X)$$

so to see that it preserves finite totalizations it suffices to show that $B(X, X)_{hC_2}$ and $\mathrm{L}_{\text{♀}}(X)$ preserve finite totalizations which follows as in the last proof.

(2)⇒(3) is clear and for (3)⇒(1) assume that ♀ sends pushouts to totalizations. Then also the cross-effect has this property in each variable separately as the formulas for it ist just compatible with totalizations. It follows that the cross-effect is bilinear by applying Lemma 3.12 in each variable separately. As before we then see

[10] This is the definition of being quadratic in the sense of Eilenberg-MacLane which makes sense for semi-additive ∞-categories

that $B(X,X)_{hC_2}$ sends reflexive coequalizers to totalizations so that also $L_{Ϙ}$ does. Moreover we have that $L_{Ϙ}$ is additive since

$$\begin{aligned} \mathrm{L}_{Ϙ}(X \oplus Y) &= \frac{Ϙ(X \oplus Y)}{B(X \oplus Y, X \oplus Y)_{hC_2}} \\ &= \frac{Ϙ(X) \oplus B(X,Y) \oplus Ϙ(Y)}{B(X,X)_{hC_2} \oplus B(X,Y) \oplus B(Y,Y)_{hC_2}} \\ &= L_{Ϙ}(X) \oplus L_{Ϙ}(Y)\ . \end{aligned}$$

Thus $L : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is exact by Lemma 3.12. □

We want to end this section by giving a 'classification' of quadratic functors.

Lemma 3.15. (1) *The functor*

$$\begin{aligned} \mathrm{Fun}^q(\mathcal{C}) &\to \mathrm{Fun}^{\mathrm{s}}(\mathcal{C}) \\ Ϙ &\mapsto B_{Ϙ} \end{aligned}$$

admits both adjoints given by $B \mapsto Ϙ^q_B$ and $B \mapsto Ϙ^s_B$ (see Example 3.7) and they are fully faithful.

(2) *The full inclusion*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \subseteq \mathrm{Fun}^q(\mathcal{C})$$

admits both adjoints. The left adjoint is given by $Ϙ \mapsto \mathrm{L}_{Ϙ}$ and the right adjoint by the fibre of $Ϙ(X) \to Ϙ^s_{B_{Ϙ}}(X) = B_{Ϙ}(X,X)^{hC_2}$.

Proof. We first observe that the functor $\mathrm{Fun}^q(\mathcal{C}) \to \mathrm{Fun}^{\mathrm{s}}(\mathcal{C})$ induces an equivalence of ∞-categories when restricted to the full subcategory of homogenous functors with inverse given by $B \mapsto Ϙ^q_B$. The assigment $\mathrm{Fun}^q(\mathcal{C}) \to \mathrm{Fun}^q(\mathcal{C})$ given by $Ϙ \mapsto Ϙ^q_{B_{Ϙ}}$ then defines an endofunctor L of Fun^q which comes with a natural transformation $L \to \mathrm{id}$ which is idempotent, i.e. the two induced maps $L^2 \to L$ are equivalence and homotopic to each other. Thus it follows that it is a colocalization (by HTT Proposition 5.2.7.4) and thus the left adjointness. A similar argument works for the right adjoint which shows the first part of the claim.

For the second part we note that we have for $Ϙ$ quadratic and L linear a fibre sequence

$$\mathrm{Map}(L_{Ϙ}, L) \to \mathrm{Map}(Ϙ, L) \to \mathrm{Map}(Ϙ^q_{B_{Ϙ}}, L)$$

from the fibre sequence $B^q_{Ϙ} \to Ϙ \to L_{Ϙ}$. Thus the claim follows from the fact that

$$\mathrm{Map}(Ϙ^q_{B_{Ϙ}}, L) \simeq \mathrm{Map}(B_{Ϙ}, B_L) = \mathrm{Map}(B_{Ϙ}, 0) = 0.$$

The right adjoint works similary, except we first have to observe that the fibre of

$$Ϙ(X) \to B_{Ϙ}(X,X)^{hC_2}$$

is indeed linear, as its cross effect vanishes. □

Remark 3.16. Such a setting, induced by an exact functor $\pi : \mathcal{D} \to \mathcal{C}$ between stable ∞-categories which admits both adjoints L and R which are fully faithful (equivalently, one of them is fully faithful) is called a *stable recollement.* Then the inclusion of the kernel $\ker(\pi) \subseteq \mathcal{C}$ also admits both adjoints given by

$$X \mapsto \mathrm{cofib}(L\pi X \to X) \qquad X \mapsto \mathrm{fib}(X \to R\pi X).$$

Moreover $\mathcal{D} \to \mathcal{C}$ is a Dwyer-Kan localization, in fact it is the Verdier Quotient $\mathcal{D}/\ker(\pi)$. We will come back to that when discussing split Verdier quotients later.

For any quadratic functor $Ϙ$ we have a pullback square (a sort of 'fracture square')

$$\begin{tikzcd} Ϙ(X) \arrow[r] \arrow[d] & L_{Ϙ}(X) \arrow[d] \\ B(X,X)^{hC_2} \arrow[r] & B(X,X)^{tC_2} . \end{tikzcd}$$

in wich the right vertical map is the map induced from the left hand map, but can also be described using that $B(X,X)^{tC_2}$ is linear. Thus any quadratic functor $Ϙ$ is determined by its cross effect $B_{Ϙ} \in \mathrm{Fun}^{\mathrm{s}}(\mathcal{C})$, its linear part $L_{Ϙ} \in \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C})$ and the natural map $L_{Ϙ}(X) \to B(X,X)^{tC_2}$, considered as a morphism in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C})$. Conversely any such triple determines a quadratic functor with the respective linear and bilinear part. We can therefore deduce the following classification of quadratic functors (which is a special case of a more general result from Goodwillie calculus):

Corollary 3.17. *The ∞-category of quadratic functors is equivalent to the ∞-category of triples $B \in \mathrm{Fun}^{\mathrm{s}}(\mathcal{C})$, $L \in \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C})$ and $L \to B^{tC_2}$ in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C})$. More precisely we have a pullback*

$$\begin{tikzcd} \mathrm{Fun}^{q}(\mathcal{C}) \arrow[r] \arrow[d, "B"] & (\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}))^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Fun}^{s}(\mathcal{C}) \arrow[r, "\_^{tC_2}"] & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}) . \end{tikzcd}$$

of ∞-categories.

4. Poincaré-∞-categories

Definition 4.1. *A symmetric bilinear functor $B \in \mathrm{Fun}^{s}(\mathcal{C})$ is* non-degenerate *if for any $Y \in \mathcal{C}$ the functor*

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Sp} \qquad X \mapsto B(X,Y)$$

is representable by an object $DY \in \mathcal{C}$, i.e. we have a natural equivalence $B(X,Y) \simeq \mathrm{map}_{\mathcal{C}}(X,Y)$.

By the stable Yoneda lemma the object DY is, if it exists, well-defined and for any Y we get a canonical map $Y \to D(DY)$ corresponding under the equivalence

$$\mathrm{map}_{\mathcal{C}}(Y, D(DY)) \simeq B(Y,DY) \simeq B(DY,Y) \simeq \mathrm{map}_{\mathcal{C}}(DY,DY)$$

to the identity.

Definition 4.2.
- *We say that $B \in \mathrm{Fun}^{s}(\mathcal{C})$ is perfect if it is non-degenerate and for every Y the map $Y \to D(DY)$ is an equivalence.*
- *We say that a quadratic functor $Ϙ : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is non-degenerate or perfect it its bilinear part is.*
- *A Poincaré ∞-category is a pair consisting of a (small) stable ∞-category $\mathcal{C}$ and a perfect quadratic functor $Ϙ : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$.*

Lemma 4.3. *For a non-degenerate symmetric bilinear functor $B \in \mathrm{Fun}^{s}(\mathcal{C})$ the duality assembles into a functor $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that we have a natural (in X and Y) equivalence*

$$B(X,Y) \simeq \mathrm{map}_{\mathcal{C}}(X,DY)$$

and D is an equivalce of ∞-categories iff B is perfect.

Proof. We observe that the stable yoneda embedding, given by the composition

$$\mathcal{C} \to \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$$

is fully faithful, since the ordinary Yoneda embedding is. As a result, postcomposition exhibits a full inclusion

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

and an object $B \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ lies in the image precisely if it does pointwise. This proves the first assertion.

For the second claim we note that the natural equivalence

$$\mathrm{Map}_{\mathcal{C}^{\mathrm{op}}}(DX, Y) = \mathrm{Map}_{\mathcal{C}}(Y, DX) = B(Y, X) \cong B(X, Y) = \mathrm{Map}_{\mathcal{C}}(X, DY) \ .$$

shows that $D^{\mathrm{op}} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is right adjoint to $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. The unit and the counit of this adjunction are both given by the map $X \to D(DX)$ considered as a transformation $\mathrm{id}_{\mathcal{C}} \to D \circ D^{\mathrm{op}}$ and $D^{\mathrm{op}} \circ D \to \mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$. Thus D is an equivalence precisely if unit and counit are equivalences which shows the claim. ☐

Example 4.4. Let R be a ring with involution σ. Then the symmetric bilinear functor $B \in \mathrm{Fun}^s(\mathcal{D}^{\mathrm{perf}} R)$ defined in 3.10 given by

$$B(X, Y) = \mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R \otimes_{\mathbb{Z}} R)}(X \otimes_{\mathbb{Z}} Y, R)$$

is perfect. To see this simply note that

$$\mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R \otimes_{\mathbb{Z}} R)}(X \otimes_{\mathbb{Z}} Y, R) \simeq \mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R)}(X, DY)$$

for $DY = \mathrm{res}_\sigma R\,\mathrm{Hom}_R(Y, R)$ and for perfect modules the functor D is an equivalence as we have seen earlier (Corollary 2.14). In particular we get that $\mathcal{C} = \mathcal{D}^{\mathrm{perf}}(R)$ with Ω^s_R and Ω^q_R form Poincaré-∞-categories.

Remark 4.5. One could actually more generally consider those object $X \in \mathcal{D}(R)$ which are reflexive instead of perfect, i.e. where $X \to D(DX)$ is an equivalence. For $R = \mathbb{Z}$ this class contains for example the non-perfect modules $\bigoplus_{n \in \mathbb{Z}} \mathbb{Z}[n]$ and $C^*(M)$ for M a space of finite type.

Example 4.6. We consider the ∞-category $\mathrm{Sp}^{\mathrm{fin}}$. This can be equipped with the following three perfect quadratic functors

$$\Omega^q_{\mathbb{S}} \to \Omega^u_{\mathbb{S}} \to \Omega^s_{\mathbb{S}}$$

all of which have the same underlying symmetric bilinear functor

$$B(X, Y) = \mathrm{map}_{\mathrm{Sp}^{\mathrm{fin}}}(X \otimes_{\mathbb{S}} Y, \mathbb{S}) = DX \otimes DY$$

with duality given by Spanier-Whitehead duality $X \mapsto DX = \mathrm{map}(X, \mathbb{S})$. The first and the last are simply induced by the symmetric bilinear functor by taking homotopy orbits and homotopy fixed points as in Example 3.7. That is we have $L_{\Omega^q} = 0$ and $L_{\Omega^s}(X) = B(X, X)^{tC_2}$.

The middle one $\Omega^u_{\mathbb{S}}$, called the *universal* quadratic functor, is more interesting. Its linear part is given by

$$L_{\Omega^u}(X) := \mathrm{map}_{\mathbb{S}}(X, \mathbb{S}) = DX \ .$$

In order to define Ω^u we thus have to provide a natural map

$$DX = L_{\Omega^u}(X) \to B(X, X)^{tC_2} = (DX \otimes_{\mathbb{S}} DX)^{tC_2} \ .$$

There is a canonical such map called the *Tate-diagonal* of DX which can be constructed as follows: such natural transformations are determined by their value on the sphere $\mathbb{S}$ by Yoneda (since the target is exact) and thus are given by maps $\mathbb{S} \to \mathbb{S}^{tC_2}$. We simply define this map as the composite

$$\mathbb{S} \xrightarrow{p^*} \mathbb{S}^{BC_2} = \mathbb{S}^{hC_2} \to \mathbb{S}^{tC_2}$$

where $p : BC_2 \to \mathrm{pt}$. Concretely the quadratic functor $Ϙ^u_{\mathbb{S}}$ is thus given by the pullback

$$\begin{tikzcd} Ϙ^u_{\mathbb{S}}(X) \arrow[r] \arrow[d] & DX \arrow[d] \\ (DX \otimes_{\mathbb{S}} DX)^{hC_2} \arrow[r] & (DX \otimes_{\mathbb{S}} DX)^{tC_2} \end{tikzcd}$$

Remark 4.7. For those who know equivariant homotopy theory: the functor $Ϙ^u$ can also be written by the genuine fixed points

$$Ϙ^u_{\mathbb{S}}(X) = (DX \otimes DX)^{C_2}$$

where we view $DX \otimes_{\mathbb{S}} DX$ as a genuine spectrum given by the Hill-Hopkins-Ravenel norm of DX (from the trivial group to C_2). Such a quadratic functor with this description also appears in work of Weiss-Williams (in a dual setting but even more generally for parametrised spectra over some base). There is also the functor

$$Ϙ^{qu}_{\mathbb{S}}(X) = D((X \otimes_{\mathbb{S}} X)^{C_2})$$

with the same bilinear part and whose linear part is given by

$$L_{Ϙ^{qu}_{\mathbb{S}}}(X) = \mathrm{map}_{\mathrm{Sp}^{\mathrm{fin}}}\left(X, \mathrm{cofib}(\mathbb{S}_{hC_2} \xrightarrow{(\mathrm{Nm}, p_*)} \mathbb{S}^{hC_2} \oplus \mathbb{S})\right) .$$

(it is much easier to describe the right adjoint linear part which is given by DX). One should think of $Ϙ^u$ as a 'genuine' version of the symmetric functor and $Ϙ^u$ as a 'genuine' version of the quadratic functor.

Example 4.8. For any given Poincaré-∞-category $(\mathcal{C}, Ϙ)$ consider the shift $Ϙ[n]$ defined as

$$Ϙ[n](X) = Ϙ(X)[n] .$$

This is again a quadratic functor whose bilinear part is given by

$$B_{Ϙ[n]}(X, Y) = B_Ϙ(X, Y)[n] = \mathrm{Map}(X, (DY)[n])$$

so that the associated duality is $D[n]$. This is also an equivalence since shifting is an equivalence. We see that $(\mathcal{C}, Ϙ[n])$ is also Poincaré.

Example 4.9. For a given stable ∞-category $\mathcal{C}$ we consider the pair $\mathrm{Hyp}(\mathcal{C}) = (\mathcal{C} \times \mathcal{C}^{\mathrm{op}}, Ϙ_{\mathrm{hyp}})$ where

$$Ϙ_{\mathrm{hyp}}(X, Y) = \mathrm{map}_{\mathcal{C}}(X, Y)$$

is the mapping spectrums functor. We claim that $\mathrm{Hyp}(\mathcal{C})$ is Poincaré. To see this we note that

$$\mathrm{map}_{\mathcal{C}}(X, Y) = \mathrm{map}_{\mathcal{C} \times \mathcal{C}^{\mathrm{op}}}((X, Y), (Y, X))_{hC_2}$$

so that $\mathrm{map}_{\mathcal{C}}$ is in fact homogenous with associated duality given by $(X, Y) \mapsto (Y, X)$.

Example 4.10. For any Poincaré-∞-category $(\mathcal{C}, \Qoppa)$ we define a pair $\mathrm{Met}(\mathcal{C}) = (\mathcal{C}^{\Delta^1}, \Qoppa_{\mathrm{met}})$ whose underlying ∞-category is the arrow category of $\mathcal{C}$ and whose quadratic functor is given by

$$\Qoppa_{\mathrm{met}}(L \to X) = \mathrm{fib}\left(\Qoppa(X) \to \Qoppa(L)\right) \ .$$

This functor is again quadratic with associated perfect duality given by

$$D_{\mathrm{met}}(L \to X) = D(X/L) \to DX \ .$$

The fact that it is quadratic simply follows from the fact that it is the fibre of quadratic functors. The cross effect can then be computed as the fibre of the cross effects:

$$\begin{aligned} B_{\mathrm{met}}(L \to X, L' \to X') &= \mathrm{fib}\big(B_{\Qoppa}(X, X') \to B_{\Qoppa}(L, L')\big) \\ &= \mathrm{fib}\big(\operatorname{map}_{\mathcal{C}}(X, DX') \to \operatorname{map}_{\mathcal{C}}(L, DL')\big) \\ &= \mathrm{fib}\Big(\operatorname{map}_{\mathcal{C}^{\Delta^1}}(L \to X, DX' \to DX') \to \operatorname{map}_{\mathcal{C}^{\Delta^1}}(L \to X, DL' \to 0')\Big) \\ &= \operatorname{map}_{\mathcal{C}^{\Delta^1}}(L \to X, \mathrm{fib}(DX' \to DL') \to DX') \\ &= \operatorname{map}_{\mathcal{C}^{\Delta^1}}(L \to X, D(X'/L') \to DX') \ . \end{aligned}$$

which shows the claim.

5. Poincaré objects

Let us recall Definition 1.9 from the Introduction:

Definition 5.1. *Let $(\mathcal{C}, \Qoppa)$ be a Poincaré-∞-category. A* Poincaré-object *(of dimension 0) in $\mathcal{C}$ is given by an object $X \in \mathcal{C}$ together with a map $q : \mathbb{S} \to \Qoppa(X)$ s.t. the induced map $\tilde{q} : X \to DX$ is an equivalence, where $\tilde{q}$ is the image of q under the map*

$$\Qoppa(X) \xrightarrow{+^*} \Qoppa(X \oplus X) = \Qoppa(X) \oplus \Qoppa(X) \oplus \operatorname{map}_{\mathcal{C}}(X, DX) \to \operatorname{map}_{\mathcal{C}}(X, DX) \ .$$

For a Poincaré-object *of dimension n we have instead a map $\mathbb{S}^n \to \Qoppa(X)$ such that the associated map $\tilde{q} : X[n] \to DX$ is an equivalence. In general we shall refer to q as a $\Qoppa$-form on X (even if it is not Poincaré).*

Remark 5.2. Let us note that a Poincaré object for $(\mathcal{C}, \Qoppa)$ of dimension n is the same as a Poincaré object of dimension 0 for the Poincaré-∞-category $(\mathcal{C}, \Qoppa[-n])$. Thus we will henceforth mostly speak about Poincaré objects and drop the dimension (which will then automatically be zero).

As indicated above, one should think of the element q as a form on X. To make this precise let us work that out in some examples.

Construction 5.3. Let R be a ring with involution σ and consider the Poincaré-∞-category $(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^s_R)$. Then for a Poincare object (X, q), or more generally any object X with a map $\mathbb{S} \to \Qoppa^s_R(X)$, we get an induced σ-symmetric, bilinear form on the R-module $H_0(X)$, that is a map

$$\beta : H_0(X) \otimes_{\mathbb{Z}} H_0(X) \to R$$

such that β is linear in the first coordinate and $\beta(x,y) = \sigma\beta(y,x)$. To see this we simply look at the induced map

$$\begin{aligned}\pi_0(\mathrm{map}(X \otimes_{\mathbb{Z}} X, R)^{hC_2}) &\to \pi_0(\mathrm{map}(X \otimes_{\mathbb{Z}} X, R))^{C_2} \\ &\xrightarrow{H_0} \mathrm{Hom}(H_0(X \otimes_{\mathbb{Z}} X), R)^{C_2} \\ &\to \mathrm{Hom}(H_0 X \otimes_{\mathbb{Z}} H_0 X, R)^{C_2}\end{aligned}$$

And the latter is the associated form. We also note that the induced map $\tilde{q} : X \to DX$ induces the respective map

$$H_0 X \to H_0(DX) \to D(H_0 X) = \mathrm{Hom}_R(H_0 X, R)$$

which might in general not be an isomorphism. But if we for example work over a field then the latter map is of course an isomorphism. So that in this case the induced σ-symmetric bilinear form on $H_0(X)$ is unimodular. Of course we could have taken H_* instead of H_0 and would get a graded σ-symmetric form in $H_*(X)$.

If $X = P[0]$ for a projective module over R (or more generally X is any connective chain complex) then all of the maps above are isomorphisms so that we get that a connective Poincaré object for $(\mathcal{D}^{\mathrm{perf}}(R), \Omega^s_R)$ is really the same as a unimoduar, σ-symmetric form on $H_0(X)$.

Similarly, if we have a Poincaré object of dimension n then we will get an associated form

$$H_* X \otimes H_*(X) \to R[-n]$$

which for example gives for n even a form on the middle dimensional homology $H_{n/2}(X)$ (there are some signs showing up here).

Example 5.4. Recall Example 1.10 from the introduction, namely that for an n-dimensional closed, oriented, compact manifold M we get that $C^*(M,R)$ is a Poincaré object of dimension n in $(\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}), \Omega^s_R)$. For $R = \mathbb{Q}$ we get an induced unimodular form on the middle dimensional homology, the intersection form.

Since every quadratic functor on $\mathcal{D}^{\mathrm{perf}}(R)$ with the standard bilinear part has a map to Ω^s_R so that we get in all cases such an induced form on the homology. But we will more closely look at the functor Ω^q_R and justify why it is called 'quadratic'. The reason is that it is related to quadratic forms on modules as we will explain now.

For simplicitly we restrict to the case of commutative rings. Recall that a quadratic form on an R-module M is given by a map (of sets)

$$q : M \to R$$

such that $q(rm) = r^2 m$ and such that $\beta(m,n) = q(m+n) - q(m) - q(n)$ is bilinear. The latter is clearly symmetric. If 2 is a unit in the ring R then we see that we we can recover q from β as $q(m) = \frac{\beta(m,m)}{2}$, so that over rings in which 2 is invertible we find that quadratic forms and symmetric bilinear forms are the same. In fact, if 2 is a unit in R (even for a ring with involution) we also get that the norm map

$$\Omega^q_R \to \Omega^s_R \ .$$

is an equivalence. We note that for any given object $X \in \mathcal{D}^{\mathrm{perf}}(R)$ and R-commutative we have a canonical map

$$\Omega^q_R(X) = \mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R)}(X \otimes_R X, R)_{hC_2} \to \mathrm{map}_{\mathcal{D}(R)}\big((X \otimes_R X)^{hC_2}, R\big)$$

which is the canonical interchange map applied to the functor $\mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R)}(-, R)$. In fact, we have the following result (which we will not really use but prove for concreteness).

Lemma 5.5. *This map is an equivalence of spectra.*

Proof. We note that both functors

$$X \mapsto \mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R)}(X \otimes_R X, R)_{hC_2}$$

and

$$X \mapsto \mathrm{map}_{\mathcal{D}(R)}(X \otimes_R X, R)_{hC_2}$$

are quadtratic with cross-effect given by the usual bilinear functor $B(X,Y) = \mathrm{map}_{\mathcal{D}^{\mathrm{perf}}(R)}(X \otimes_R X, R)_{hC_2}$. Thus we only have to compare the linear parts. But the linear parts will agree on $\mathcal{D}^{\mathrm{perf}}(R)$ if they agree for $X = R$ so that it suffices to check that the map above is an equivalence for $X = R$. But there the map is given by the canonical map

$$R_{hC_2} \to \mathrm{map}_{\mathcal{D}(R)}(R^{hC_2}, R)$$

from the R-homology of BC_2 to the dual of the cohomology, which is an equivalence since BC_2 is of finite type (in fact, one can also simply compute both sides). $\square$

Now for a given Poincaré object (X, q) in $(\mathcal{D}^{\mathrm{perf}}(R), \Omega^q_R)$ we consider the associated map

$$(X \otimes_R X)^{hC_2} \to R$$

which thus gives rise to a map obtained by applying H_0

$$H_0\Big((X \otimes_R X)^{hC_2}\Big) \to R\ .$$

Lemma 5.6. *For any X the assignment*

$$q : H_0(X) \to H_0\Big((X \otimes_R X)^{hC_2}\Big)$$

given by the composition

$$\Omega^\infty X \xrightarrow{\Delta} (\Omega^\infty X \times \Omega^\infty)^{hC_2} \to \Omega^\infty\big((X \otimes_R X)^{hC_2}\big)$$

is a quadratic form on $H_0(X)$ with values in $H_0\Big((X \otimes_R X)^{hC_2}\Big)$.

Proof. We have to show that $q(rx) = r^2 q(x)$ for $r \in \pi_0(R)$ and that

$$\beta(x,y) = q(x+y) - q(x) - q(y)$$

is bilinear.

The map $\Omega^\infty X \to \Omega^\infty\,(X \otimes_R X)^{\mathrm{hC}_2}$ is natural in X. In particular for every $r \in \Omega^\infty R$ the morphism $l_r : X \to X$ obtained by left multiplication by r the diagram

$$\begin{array}{ccc}
\Omega^\infty X & \longrightarrow & \Omega^\infty\,(X \otimes_R X)^{hC_2} \\
\Big\downarrow{\scriptstyle \Omega^\infty l_r} & & \Big\downarrow{\scriptstyle \Omega^\infty (l_r \otimes l_r)^{hC_2}} \\
\Omega^\infty X & \longrightarrow & \Omega^\infty\,(X \otimes_R X)^{hC_2}
\end{array}$$

commutes. But by bilinearity the right vertical map is equivalent to left multiplication with r^2 on the R-module $(X \otimes_R X)^{hC_2}$. Upon applying π_0 this implies the equality $q(rx) = r^2 q(x)$.

Similarly, applying naturality for the fold map $+ :: X \oplus X \to X$ we find a commutative square

$$\begin{tikzcd}
\Omega^\infty(X \oplus X) \arrow[r] \arrow[d, "\Omega^\infty +"] & \Omega^\infty\big((X \oplus X) \otimes_R (X \oplus X)\big)^{hC_2} \arrow[d, "\Omega^\infty(+\otimes+)^{hC_2}"] \\
\Omega^\infty X \arrow[r] & \Omega^\infty \left(X \otimes_R X\right)^{hC_2} .
\end{tikzcd}$$

Under the distributivity equivalence

$$\big((X \oplus X) \otimes_R (X \oplus X)\big)^{hC_2} \simeq (X \otimes_R X)^{hC_2} \oplus (X \otimes_R X) \oplus (X \otimes_R X)^{hC_2}$$

the right vertical map $(+ \otimes +)^{hC_2}$ in this square is given by $(\mathrm{id}, \mathrm{Nm}, \mathrm{id})$ where Nm is the norm map $X \otimes_R X \to (X \otimes_R X)^{hC_2}$. Thus applying π_0 we get the identity

$$q(x + y) = q(x) + (\pi_0\mathrm{Nm})(x \otimes y) + q(y)$$

or equivalently $\beta(x, y) = (\pi_0\mathrm{Nm})(x \otimes y)$. But Nm is R-linear which implies the claim.

$\square$

Proposition 5.7. *For every element $\mathbb{S} \to \Qoppa^q_R(X)$ with $X \in \mathcal{D}^{\mathrm{perf}}(R)$ we get an induced quadratic form on $H_0(X)$. If $X = P[0]$ for P f.g. projective then this induces an isomorphism*

$$\pi_0\Qoppa^q_R(X) \to \{ \text{ Quadratic forms on } P \text{ } \} \ .$$

Proof. The first claim just follows from the construction given before. For the latter we note that if P is projective then also $P \otimes_R P$ is projective and we have that

$$\mathrm{map}_{\mathcal{D}(R)}(P \otimes_R P, R) = \mathrm{Hom}_R(P \otimes_R P, R)$$

is concentrated in degree zero and given by the abelian group of bilinearforms on P withe the C_2-action by flipping. Thus we find that

$$\pi_0\Qoppa^q_R(P[0]) = \mathrm{Hom}_R(P \otimes_R P, R)_{C_2} = \{\text{Bilinear forms on } P\}_{C_2}$$

Unwinding the constructions we see that the map in question from the set of bilinear forms to quadratic forms sends a bilinear form γ to the associated quadratic form

$$q_\gamma(X) = \gamma(x, x) \ .$$

Thus to show the claim we have to verify that every quadratic form on P is of this form and that two bilinear forms γ and γ' give rise to the same quadratic form precisely if the represent the same orbit.

We can assume without loss of generality that P is free and then represent bilinear forms by matrices. For the last assertion assume that γ is such that

$$\gamma(x, x) = 0$$

for all $x \in P$. This means that γ is represented by a skew symmetric matrix. But then we can write it as the antisymmetrization of a bilinear form, e.g. the upper triangular part. This shows that

$$\{\text{Bilinear forms on } P\}_{C_2} \to \{ \text{ Quadratic forms on } P \text{ } \}$$

is injective. For surjectivity we have to argue that every quadratic form is obtained from a bilinear form γ represented by a matrix A. To see this consider arbitrary q and let β be its polarization represented by a symmetric matrix B. We now set

A to be the matrix whose upper triangular part agrees with B and which on the diagonal is given by q. Then we have that $A + A^T = B$, thus the quadratic form q_γ has the property that its associated bilinear form agrees with q on a basis and that the polarizations agree. Thus by the formula for polarizations they agree. $\square$

Remark 5.8. It is an interesting fact, due to Eilenberg-MacLane, which uses similar methods, that for an abitrary abelian groups M we have that

$$H^4(K(A,2),\mathbb{Z}) = \{\text{quadratic forms on } A\}.$$

One can also show that this group is isomorphic to $\mathrm{Hom}((A\otimes A)^{C_2},\mathbb{Z})$ for any abelian group A and not just projective ones. We do not know however if over arbitrary rings R and arbitrary modules M the canonical map

$$\mathrm{Hom}((M\otimes_R M)^{C_2}, R) \to \{R\text{-linear quadratic forms on } M\}.$$

is an isomorphism. For $M = P$ finitely generated projective it follows form the previous result (with some translations).

Remark 5.9. If one works with a ring with involution (R,σ) instead of commutative rings then one gets for every $\Qoppa^q_R$-form on X a quadratic form on $\pi_0(X)$ with values in $R/(x-\sigma x)$.

Remark 5.10. There was a very intersting question by Søren Galatius: what is $\pi_1\Qoppa^q_R(P[0])$ for P finitely generated projective, i.e. when we know that π_0 is given by quadratic forms. We then find that this is given by the second group homology of C_2-with values in the module of bilinear forms on P. This is given by the kernel of the map $1-\sigma$ module the cokernel of $1+\sigma$, thus by the Quotient

$$\pi_1\Qoppa^q_R(P[0]) = \frac{\{\text{Symmetric bilinear forms on } P\}}{\{\text{Quadratic forms on } P\}}\ .$$

In fact every odd homotopy group is isomorphic to that and the even ones, except for the zero'th are given by

$$\pi_{2n}\Qoppa^q_R(P[0]) = \frac{\{\text{Antisymmetric bilinear forms on } P\}}{\{\text{Antisymmetrizations of bilinear forms}\}} = 0$$

We do not really know what this 'means'.

Example 5.11. A Poincaré object in $\mathrm{Hyp}(\mathcal{C})$ is the same as a pair of objects (X,Y) of $\mathcal{C}$ together with an equivalence $X \to Y$. This is essentially the same as an object $X \in \mathcal{C}$.

Now after shedding some light on the functors $\Qoppa^q_R$ and $\Qoppa^s_R$ we shall start to develop some theory.

Construction 5.12.

(1) For a pair (X,q) and (X',q') of Poincaré objects we consider the direct sum $X\oplus X'$ equipped with the form $q+q' \in \Qoppa(X\oplus X')$ induced under the canonical map $\Qoppa(X)\oplus\Qoppa(X') \to \Qoppa(X\oplus X')$. The induced map is

$$\widetilde{q+q'} = \tilde{q} + \tilde{q'} : X\oplus X' \to DX \oplus DX'$$

so that it is also Poincaré.

(2) For a Poincaré object (X,q) we have the Poincaré object $(X,-q)$ with adjunct map $X \to DX$ given by $-\tilde{q}$.

(3) For a given Poincaré-∞-category $(\mathcal{C}, ♀)$ and $X \in \mathcal{C}$ we consider
$$\mathrm{hyp}(X) := X \oplus DX$$
with the form q given by the image of the identity under the map
$$\mathrm{map}_{\mathcal{C}}(X, X) = B(X, DX) \to ♀(X \oplus DX). \tag{1}$$
We find that the adjunct map $\tilde{q} : X \oplus DX \to DX \oplus X$ is given by the usual hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which can be seen as follows: the map (1) can also be factored as
$$\begin{aligned} B(X, DX) &\to B(X, X)_{hC_2} \oplus B(X, DX) \oplus B(DX, DX)_{hC_2} \\ &= B(X \oplus DX, X \oplus DX)_{hC_2} \\ &\to ♀(X \oplus DX) \end{aligned}$$
so that postcomposing the map (1) with the map
$$♀(X \oplus DX) \to B(X \oplus DX, X \oplus DX)^{hC_2}$$
is simply the norm.

Definition 5.13. *A Lagrangian (aka nullbordism) for an n-dimensional Poincaré object (X, q) in $(\mathcal{C}, ♀)$ is a pair consisting of a map $L \to X$ together with a path connecting $q|_L$ to 0 such that the sequence*
$$L[n] \to X[n] \cong DX \to DL$$
with the induced nullhomotopy of the composite is a fibre sequence. In this case the object (X, q) is called metabolic.

Lemma 5.14. *Metabolic Poincaré objects are essentially the same as Poincaré objects in* $\mathrm{Met}(\mathcal{C}, ♀)$.

Proof. We treat the case of dimension 0. A $♀_{\mathrm{met}}$-form in $\mathrm{Met}(\mathcal{C})$ on an object $L \to X$ is simply given by a $♀$-form on on the X and a nullhomotopy of the restriction to L. Then we simply have to check that the induced map form $L \to X$ into the dual $D(L/X) \to DX$ is an equivalence. But this simply means that X is a Poincaré object (i.e. $X \simeq DX$) and that the L is a Lagrangian since $L \simeq D(X/L)$ is equivalent to $X/L \simeq DL$. □

We finish this section by giving a definition of maps between Poincaré-∞-categories. This are rigged to preserve Poincaré objects.

Definition 5.15. *A map of Poincaré categories $(\mathcal{C}, ♀)$ and $(\mathcal{C}', ♀')$ consists of an exact functor $F : \mathcal{C} \to \mathcal{C}'$ together with a natural transformation $\eta : ♀ \to ♀' \circ F$ such that for every object $X \in \mathcal{C}$ a certain map*
$$F(DX) \to D(FX)$$
induced from η is an equivalence in $\mathcal{C}$. This map is the image of the identity under the map
$$\mathrm{map}_{\mathcal{C}}(DX, DX) = B_{♀}(X, X) \xrightarrow{B_\eta} B_{♀'}(F(DX), FX) = \mathrm{map}_{\mathcal{C}'}(F(DX), D(FX)) \ .$$

Clearly we find that for a Poincaré object (X, q) in $(\mathcal{C}, ♀)$ the induced object $(FX, \eta(q))$ is also Poincaré, one only has to check that the equivalence $F(DX) \cong D(FX)$ is compatible with adjunct maps.

Example 5.16. For a ring with involution we can consider the map

$$(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^q_R) \to (\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^s_R)$$

given by the identity functor and the transformation η induced by the map $\Qoppa^q_R \to \Qoppa^s_R$. This preserves the duality (in fact is an equivalence on bilinear parts). The assignment on Poincaré objects is the assignment of symmetric forms to quadratic forms. More generally maps of Poincaré categories that are the identity on objects always have to preserve the bilinear part and thus can only change the linear part.

Example 5.17. For a given Poincaré-∞-category $\mathcal{C}$ there is a functor

$$\mathrm{Hyp}(\mathcal{C}) \to \mathcal{C} \qquad (X, Y) \mapsto X \oplus DY \ .$$

This comes with a natural transformation

$$\eta : \Qoppa_{\mathrm{hyp}}(X, Y) = \mathrm{map}_{\mathcal{C}}(X, Y) \to \Qoppa(X) \oplus \mathrm{map}_{\mathcal{C}}(X, Y) \oplus \Qoppa(DY) = \Qoppa(X \oplus DY)$$

and the induced map is the equivalence $DX \oplus Y \simeq D(X \oplus DY)$. The induced map on Poincaré objects sends $X \in \mathcal{C}$ to the hyperbolic object $\mathrm{hyp}(X)$.

Example 5.18. $\mathrm{Met}(\mathcal{C}, \Qoppa) \to \mathcal{C}$ given by evaluation at the target refines to a functor of Poincaré objects. The induced map sends $L \to X$ simply to X. There is also a functor

$$\mathrm{Hyp}(\mathcal{C}) \to \mathrm{Met}(\mathcal{C}) \qquad (X, Y) \mapsto (X \to X \oplus DY)$$

and we have that

$$\Qoppa_{\mathrm{met}}(X \to X \oplus DY) = \mathrm{fib}(\Qoppa(X \oplus DY) \to \Qoppa(X)) = \mathrm{map}_{\mathcal{C}}(X, Y) \oplus \Qoppa(DY) \ .$$

We have an inclusion $\Qoppa_{\mathrm{hyp}}(X, Y) \to \Qoppa_{\mathrm{met}}(X \to X \oplus DY)$ and this defines a map of Poincaré-∞-categories. On Poincaré objects this simply sends an object $X \in \mathcal{C}$ to the associated hyperbolic object, considered with its canonical Lagrangian.

Proposition 5.19. *A Poincaré object in $(\mathcal{C}, \Qoppa)$ is essentially the same as a map $(\mathrm{Sp}^{\mathrm{fin}}, \Qoppa^u) \to (\mathcal{C}, \Qoppa)$ of Poincaré-∞-categories.*

Proof. We first construct a 'universal' Poincaré object in $(\mathrm{Sp}^{\mathrm{fin}}, \Qoppa^u)$: this has underlying object $\mathbb{S} \in \mathrm{Sp}^{\mathrm{fin}}$ and the form is given by an element in $\Qoppa^u(\mathbb{S})$. To this end we recall that $\Qoppa^u(\mathbb{S})$ is given by the pullback

$$\begin{tikzcd}
\Qoppa^u(\mathbb{S}) \arrow[r] \arrow[d] & D\mathbb{S} \arrow[d, "\mathrm{triv}"] \\
(D\mathbb{S})^{hC_2} \arrow[r, "\mathrm{can}"] & (D\mathbb{S})^{tC_2}
\end{tikzcd} \ .$$

There is a canonical map $q^u : \mathbb{S} \to \Qoppa^u(\mathbb{S})$ given by the identity on the right upper factor and the pullback $\mathbb{S} \xrightarrow{p^*} D\mathbb{S}^{hC_2}$ on the left left, which fit together by construction of the right hand map. [11] The underlying bilinear form of element is represented by the trivial map on the sphere $\mathbb{S} \otimes \mathbb{S} \to \mathbb{S}$ thus non-degenerate so that we see that $(\mathbb{S}, q^u)$ is Poincaé.

Now for any map $(\mathrm{Sp}^{\mathrm{fin}}, \Qoppa^u) \to (\mathcal{C}, \Qoppa)$ we get an induced Poincaré object in $(\mathcal{C}, \Qoppa)$ by pushing the universal object $(\mathbb{S}, q^u)$ forward. We need to verify that this assignment

[11] One can in fact show that $\Qoppa^u(\mathbb{S}) = \mathbb{S}_{hC_2} \oplus \mathbb{S}$ and under this equivalence the element in question is given by $(0, 1)$.

induces a 1-1 correspondence on equivalence classes. To this end let us analyse more precisely what it means to give map

$$(\mathrm{Sp}^{\mathrm{fin}}, \Qoppa^u) \to (\mathcal{C}, \Qoppa) \ .$$

First of all, an exact functor $\mathrm{Sp}^{\mathrm{fin}} \to \mathcal{C}$ is unqiuely determined by its value on $\mathbb{S}$ and this gives a correspondence between objects of $\mathcal{C}$ and exact functors $\mathrm{Sp}^{\mathrm{fin}} \to \mathcal{C}$. For such a given functor $F : \mathrm{Sp}^{\mathrm{fin}} \to \mathcal{C}$ we consider the pulled back quadratic functor $\Qoppa' := F^*\Qoppa$ on $\mathrm{Sp}^{\mathrm{fin}}$ and analyse what it means to give a transformation $\eta : \Qoppa^u \to \Qoppa'$. By our decomposition such a transformation is given by a map $D \to L'_{\Qoppa} = F^* L_{\Qoppa}$ and a map of bilinear functors $D \otimes D \to B'_{\Qoppa} = F^* B_{\Qoppa}$ together with a homotopy on the respective Tate terms.

Now we use the following facts:

(1) for every exact functor $L : (\mathrm{Sp}^{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{Sp}$ a transformation $D \to L$ is determined on $\mathbb{S}$ (i.e. the spectrum of maps $D \to L$ is equivalent to $L(\mathbb{S})$).
(2) For a bilinear functor $B : (\mathrm{Sp}^{\mathrm{fin}})^{\mathrm{op}} \times (\mathrm{Sp}^{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{Sp}$ a transformation $D \otimes D \to B$ is determined on $(\mathbb{S}, \mathbb{S})$ (i.e. the spectrum of such is equivalent to $B(\mathbb{S}, \mathbb{S})$.

Thus we find that our transformation is just a triple given by an element in $L'_{\Qoppa}(\mathbb{S}) = L_{\Qoppa}(C)$, an element in the homotopy fixed points $B'_{\Qoppa}(\mathbb{S}.\mathbb{S}) = B_{\Qoppa}(C, C)$ and a path in $B_{\Qoppa}(C, C)^{tC_2}$. But this then exactly assembles to an element in $\Qoppa'(\mathbb{S}) = \Qoppa(C)$. Thus we find that transformations $\eta : \Qoppa^u \to F^*\Qoppa$ are precisely given by elements in $\Qoppa(F(\mathbb{S}))$, i.e. $\Qoppa$-forms on $F(\mathbb{S})$. Then the non-degeneracy precisely works out to show that this functor is Poincaré if the form on $F(\mathbb{S})$ is. □

Remark 5.20. One could also work with dg-categories $\mathcal{C}$ and dg-functors $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$ instead of stable ∞-categories and functors to spectra throughout.[12] It was a question of I. Patchkoria which object $(\mathcal{C}, \Qoppa)$ is 'universal', i.e. plays the role of $(\mathrm{Sp}^{\mathrm{fin}}, \Qoppa^u)$ in this world. The answer is unclear.

6. L-groups and the Grothendieck–Witt group

Definition 6.1. *Let $(\mathcal{C}, \Qoppa)$ be a Poincaré-∞-category. The L-groups of $(\mathcal{C}, \Qoppa)$ are defined as the abelian groups*

$$L_n(\mathcal{C}, \Qoppa) = \frac{\{\textit{Iso. classes of } n\textit{-dimensional Poincaré objects in } (\mathcal{C}, \Qoppa)\}}{\{\textit{metabolic Poincaré objects}\}}$$

where this quotient is taken in abelian monoids (under direct sum). We also set for a ring with involution $L^s_*(R, \sigma) := L_*(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^s)$ and $L^q_*(R, \sigma) := L_*(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^q)$ and call these the quadratic *and* symmetric *L-groups of R.*

Remark 6.2. A word of warning is in order. This is highly non-standard notation. Firstly, Lurie writes $\mathrm{L}_n(R, \sigma)$ for the L-theory of the category $\mathcal{D}^{\mathrm{fp}}(R)$, which is usually referred to as free L-theory, whereas we use the category $\mathcal{D}^{\mathrm{perf}}(R)$ which produces what is called projective L-theory. In the case where the algebraic K-theory $\mathrm{K}_0(R) \cong \mathbb{Z}$ via the canonical map $\mathbb{Z} \to R$, there is no difference between the two constructions. For our purposes it will be much better to study the projective L-groups rather than the free ones. Secondly, this notation differs from the notation introduced by Ranicki (which is the standard reference for algebraic L-theory). He

[12]Equivalently: $H\mathbb{Z}$-linear stable ∞-categories $\mathcal{C}$ and $H\mathbb{Z}$-linear functors $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$.

writes $\mathrm{L}^\bullet(R,\sigma)$ for symmetric L-theory and $\mathrm{L}_\bullet(R,\sigma)$ for quadratic L-theory and usually also means free L-theory.

Proposition 6.3. *The abelian monoids $L_n(\mathcal{C},\Omega)$ are groups, the inverse of (X,q) is given by $(X,-q)$. Moreover these groups are functorial in maps of Poincaré-∞-categories.*

Proof. We have to show that $(X,q)+(X,-q)$ admits a Lagrangian. Such a Lagrangian is given by the diagonal subspace

$$X \xrightarrow{\Delta} X \oplus X$$

together with a certain nullhomotopy of $\Delta^*(q \oplus -q)$. To construct this nullhomotopy observe that the composite

$$\Omega(X) \oplus \Omega(X) \to \Omega(X \oplus X) \xrightarrow{\Delta^*} \Omega(X)$$

is the identity on each summand, thus given by the addition of the spectrum $\Omega(X)$. Therefore $\Delta^*(q \oplus -q) = q - q = 0$ comes with a canonical nullhomotopy. For this nullhomotopy the induced homotopy in the sequence

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{\tilde{q} \oplus -\tilde{q}} DX \oplus DX \xrightarrow{\nabla} DX$$

is given by the canonical nullhomotopy, so that this becomes a fibre sequence.

The functoriality in maps of Poincaré categories is clear since those preserve Poincaré objects, sums of Poincaré objects and metabolic Poincaré objects. □

Proposition 6.4. *The canonical map $(\mathcal{D}^{\mathrm{perf}}(R), \Omega^q_R) \to (\mathcal{D}^{\mathrm{perf}}(R), \Omega^s_R)$ induces a map of abelian groups*

$$L^q_*(R) \to L^s_*(R)$$

which is an isomorphism if $\frac{1}{2} \in R$. For a general ring R the kernel and cokernel of this map are 8-torsion, in particular the map becomes an isomorphism after inverting 2 in the L-groups.

Proof. If $\frac{1}{2} \in R$ then the map $\Omega^q_R \to \Omega^s_R$ is an equivalence, as the cofibre is given by $B(X,X)^{tC_2}$ which is a module over $R^{tC_2} = 0$. The last assertion will follow later, basically by checking it for $R = \mathbb{Z}$ and adhering to multiplicative properties as we will discuss soon. □

Proposition 6.5. *The L-groups $L^q_*(R)$ and $L^s_*(R)$ are 4-periodic for any ring with involution and 2-periodic if R is an $\mathbb{F}_2$-algebra.*

Proof. We consider the shift map

$$[2] : \mathcal{D}^{\mathrm{perf}}(R) \to \mathcal{D}^{\mathrm{perf}}(R)$$

and claim that it extends to an equivalence of Poincaré-∞-categories $(\mathcal{D}^{\mathrm{perf}}(R), \Omega^s_R) \to (\mathcal{D}^{\mathrm{perf}}(R), \Omega^s_R)$ and similar for the quadratic case. To this end we observe that

$$\begin{aligned}
\Omega^s_R(X[2]) &= B\left(X[2], X[2]\right)^{hC_2} \\
&= \mathrm{map}_{\mathcal{D}(R \otimes_{\mathbb{Z}} R)}(X[2] \otimes_{\mathbb{Z}} X[2], R)^{hC_2} \\
&= \mathrm{map}_{\mathcal{D}(R \otimes_{\mathbb{Z}} R)}((X \otimes_{\mathbb{Z}} X)[4], R)^{hC_2} \\
&= \mathrm{map}_{\mathcal{D}(R \otimes_{\mathbb{Z}} R)}((X \otimes_{\mathbb{Z}} X), R)^{hC_2}[4]
\end{aligned}$$

where we have used that C_2-equivariantly $X[2] \otimes_{\mathbb{Z}} X[2] \simeq (X \otimes_{\mathbb{Z}} X)[4]$ which follows since the flip action on $\mathbb{S}^2 \otimes \mathbb{S}^2$ is trivial in homology. This is not true for the flip action on $\mathbb{S}^1 \otimes \mathbb{S}^1$ is trivial in homology, as it acts by a sign, but it is true in $\mathbb{F}_2$-homology. □

Warning 6.6. *The last two results completely fail for ring spectra in place of rings: in this generality neither the L-groups are not necessarily periodic (implied by computation of Weiss-Williams) and the map*

$$L^q_*(R)[\tfrac{1}{2}] \to L^s_*(R)[\tfrac{1}{2}] \ .$$

is not an equivalence in general. The 4-periodicity of L-groups is true however for complex orientable ring spectra R (say commutative with identity involution).

Remark 6.7. The proof of Proposition 6.5 shows that the group $L^s_{n+2}(R)$ can be described as the L-group L_n of the Poincaré-∞-category $(\mathcal{D}^{\mathrm{perf}}(R), \Omega^s_{R,-})$ where $\Omega^s_{R,-}$ is the quadratic functor given by

$$\Omega^s_{R,-}(X) = \mathrm{map}_{\mathcal{D}(R \otimes_{\mathbb{Z}} R)}((X \otimes_{\mathbb{Z}} X) \otimes_{\mathbb{Z}} \mathbb{Z}^\sigma, R)^{hC_2}$$

where $\mathbb{Z}^\sigma$ denotes the integers with the C_2-action given by the sign. This spectrum is the spectrum of antisymmetric bilinear forms and the corresponding L-groups are denotes as $L^{-s}_*(R)$. A similar observation applies to the quadratic case and yields an isomorphism $L^q_{*+2}(R) = L^{-q}_*(R)$.

Recall that we have shown that the Poincaré objects $C^*(M)$ for a closed oriented manifold M is metabolic if M is a boundary, thus the respective element in the L-group $L^s_n(\mathbb{Z})$ is trivial. We thus get a graded group homomorphism

$$\Omega^{\mathrm{SO}}_* \to L^s_*(\mathbb{Z})$$

where Ω^{SO}_* is the oriented bordism ring. This is called the Ranicki–Sullivan orientation (or genus). We will see that the groups $L^s_*(\mathbb{Z})$ are given by

$$L^s_*(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 4n, n \in \mathbb{Z} \\ \mathbb{Z}/2 & \text{for } * = 4n+1, n \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$$

and the the Sullivan-Ranicki orientation map is given by the signature of a manifold in degrees $4n$ and by the deRham invariant in degrees $4n+1$. The de Rham invariant assigns to a manifold M of dimension $(4n+1)$ the mod 2 reduction of the dimension of the 2-torsion in $H_{2n}(M)$.[13]

Proposition 6.8. *There are group homomorphisms*

$$W^s_0(R) \to L^s_0(R) \qquad W^q_0(R) \to L^q_0(R)$$

for any ring with involution where $W^s_0(R)$ and $W^q_0(R)$ are the Witt groups of symmetric, respective quadratic unimodular forms on finitely generated, projective R-modules.

Proof. Clear. □

[13]This is also given by the Stiefel-Whitney number $w_2 w_{4n-1}$ or by the Kervaire semi-characteristic

$$k(M) = \sum_{i=0}^{2n} \dim H^{2i}(M, \mathbb{R}) \mathrm{mod} 2$$

of M.

We will show in the next section that the map $W_0^q(R) \to L_0^q(R)$ is an isomorphism by means of algebraic surgery. The first map is in general not an isomorphism, but it is if R is a Dedekind ring or if $\frac{1}{2} \in R$ since then symmetric and quadratic Witt and L-groups agree.

Recall the definition of K-theory of a (small) stable ∞-category $\mathcal{C}$: the group $K_0(\mathcal{C})$ is defined as

$$K_0(\mathcal{C}) = \frac{\{\text{Isomorphism classes of objects in } \mathcal{C}\}}{[X] = [L \oplus X/L] \text{ for } L \to X \text{ a map}} \ .$$

Here the quotient is taken in abelian monoids where the monoid structure is given by direct sum, i.e. $[X] + [Y] = [X \oplus Y]$. The claim is that this monoid is already a group with inverse given by $-[X] = [X[1]]$. To see this one simply observes that we have

$$X + X[1] = X \oplus X[1] = X \oplus 0/X = 0$$

in $K_0(\mathcal{C})$.

Definition 6.9. *Let $(\mathcal{C}, Ϙ)$ be a Poincaré-∞-category. Then we define an abelian group*

$$\mathrm{GW}_0(\mathcal{C}, Ϙ) = \frac{\{\textit{Iso. classes of } 0\textit{-dimensional Poincaré objects in } (\mathcal{C}, Ϙ)\}}{[X] = [\mathrm{hyp}(L)] \textit{ for } L \to X \textit{ a Lagrangian}}$$

where the quotient is taken in abelian monoids.

Lemma 6.10. *The abelian monoid $\mathrm{GW}_0(\mathcal{C}, Ϙ)$ is an abelian group and we have a well-defined group homomorphism*

$$\mathrm{hyp} : K_0(\mathcal{C}) \to \mathrm{GW}_0(\mathcal{C}, Ϙ) \qquad [X] \mapsto [\mathrm{hyp}(X)]$$

Proof. We first prove the second claim. Thus we have to verify that for a fibre sequence

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

in $\mathcal{C}$ the relation $\mathrm{hyp}(Y) = \mathrm{hyp}(X) + \mathrm{hyp}(Z)$ holds in the monoid $\mathrm{GW}_0(\mathcal{C}, Ϙ)$. To see this note that $\mathrm{hyp}(X) + \mathrm{hyp}(Z) = \mathrm{hyp}(X \oplus Z) = \mathrm{hyp}(X \oplus DZ)$ and that the inclusion

$$X \oplus DZ \to \mathrm{hyp}(Y) = Y \oplus DY$$

extends to a Lagrangian (exercise).

To see that it is a group we claim that the inverse of (X, q) is given by $(X, -q) + \mathrm{hyp}(X[1])$. This follows since

$$[X, q] + [X, -q] = [X \oplus X, q \oplus -q] \sim \mathrm{hyp}(X)$$

where the latter equivalence follows since the diagonal is a Lagrangian as we have seen in the proof of Proposition 6.3 . Now by the previous discussion $\mathrm{hyp}(X[1])$ is an inverse to $\mathrm{hyp}(X)$. □

Lemma 6.11. *A Poincaré object X represents 0 in $\mathrm{L}_0(\mathcal{C}, Ϙ)$ precisely if it is metabolic. It represents zero in $\mathrm{GW}_0(\mathcal{C}, Ϙ)$ precisely if there are metabolic objects A, B such that*

$$X \oplus A \oplus \mathrm{hyp}(L_A[1]) \cong B \oplus \mathrm{hyp}(L_B[1]) \ .$$

Proof. For the first statement we claim that two objects X and Y in $L_0(\mathcal{C}, \Omega)$ are equivalent precisely if there is an algebraic cobordism, that is a Lagrangian in $X \oplus \overline{Y}$ (the latter denotes Y with the inverted form $-q_Y$). To see this one simply has to observe that this is actually an equivalence relation (in particular transitive). The key is to 'compose' algebraic cobordisms, which can be done be means of a pullback

$$\begin{tikzcd} & & L'' \arrow[dl] \arrow[dr] & & \\ & L \arrow[dl] \arrow[dr] & & L' \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

and we leave it to the reader to verify that L'' is a Lagrangian in $X \oplus \overline{Z}$ in a canonical way.

For the second statement we claim that we have a presentation as abelian monoids

$$\mathrm{GW}_0(\mathcal{C}, \Omega) = \frac{\{\text{Iso. classes of 0-dimensional Poincaré objects in } (\mathcal{C}, \Omega)\}}{[X] + [\mathrm{hyp}(L[1])] \text{ for } L \to X \text{ a Lagrangian}} .$$

This follows from the facts: in our definition of $\mathrm{GW}_0(\mathcal{C}, \Omega)$ we clearly have that $[X] + [\mathrm{hyp}(L[1])] = 0$. Conversely one easily sees that in the presentation above we have that $\mathrm{hyp}(L)$ and $\mathrm{hyp}(L[1])$ are inverse to one another. As a result we get that the relation $[X] = \mathrm{hyp}(L)$ holds as well.

Then finally the set of all $[X] + [\mathrm{hyp}(L[1])]$ is a submonoid of the monoid of all iso classes so that the result follows by the usual way quotients of monoids are formed. □

Proposition 6.12. *For a Poincaré ∞-category $(\mathcal{C}, \Omega)$ there is a C_2-action on $K_0(\mathcal{C})$ given by $X \mapsto DX$ and we have an exact sequence*

$$L_1(\mathcal{C}, \Omega) \xrightarrow{\mathrm{fgt}} K_0(\mathcal{C})_{C_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}_0(\mathcal{C}, \Omega) \to L_0(\mathcal{C}, \Omega) \to 0$$

Proof. The well-definedness of the C_2-action is clear, since $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is an exact functor, so that for $X \to Y \to Z$ the resulting sequence $DZ \to DY \to DX$ is also exact and thus the relation defining K-theory is preserved. To see that the first map is well-defined we have to note that if a Poincaré object $X \in L_1$ (we skip the $(\mathcal{C}, \Omega)$ for the rest of the proof to simplify notation) is metabolic, then $[X] = 0$ in $(K_0)_{C_2}$. But being metabolic for X means that we have a Lagrangian $L \to X$, in particular an exact sequence

$$L \to X \to DL[-1]$$

and thus in K_0 that $[X] = [L \oplus DL[-1]] = [L] - [DL]$ thus this is zero in the orbits.

Now to the exactness: the surjectivity on the right is clear. To see the exactness at GW_0 we first note that the composite

$$(K_0)_{C_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}_0 \to L_0$$

is obviously zero. Moreover by the previous Lemma $[X]$ is zero in L_0 iff it admits a Lagrangian L, so that in GW_0 we have

$$[X] = [\mathrm{hyp}(L)]$$

which lies in the image of the hyperbolic map.

It remains to show the exactness at $(K_0)_{C_2}$. To this end we first note that the composition

$$L_1 \xrightarrow{\text{fgt}} (K_0)_{C_2} \xrightarrow{\text{hyp}} \mathrm{GW}_0$$

is zero since for a given $X \in L_1$ we have that X can be interpreted as a Lagrangian of 0 (as a 0-dimensional Poincaé object) and thus in GW_0 we have that

$$\mathrm{hyp}(X) = 0.$$

As a result we get an induced map

$$(K_0)_{C_2}/L_1 \xrightarrow{\text{hyp}} \mathrm{GW}_0$$

and we want to show that it is injective. To this end assume that $[X]$ lies in the kernel. Then $\mathrm{hyp}(X)$ represents zero in GW_0 so that we find metabolic A and B with

$$\mathrm{hyp}(X) \oplus A \oplus \mathrm{hyp}(L_A[1]) \cong B \oplus \mathrm{hyp}(L_B[1]) \ .$$

Then this object has two different Lagrangians, namely $X \oplus L_A \oplus L_A[1]$ and $L_B \oplus L_B[1]$. The next Lemma implies that we have that the classes of these objects agree in $K_0(\mathcal{C}, ♀)_{C_2}/L_1(\mathcal{C}, ♀)$ which then shows that we have there

$$[X] = [X] + [L_A] + [L_A[1]] = [L_B] + [L_B[1]] = 0 \ .$$

This finishes the proof. □

Lemma 6.13. *For a given metabolic object X with two different Lagrangians L_1, L_2 we have that $[L_1] = [L_2]$ in $K_0(\mathcal{C}, ♀)_{C_2}/L_1(\mathcal{C}, ♀)$.*

Proof. To see this note that for two such Lagrangians we can form the pullback

$$L_1 \times_X L_2$$

and this is canonically a Poincaré object of dimension 1 (exercise, think of composition of nullbordisms). Thus from the pullback square we get in $(K_0)_{C_2}/L_1$ the relation

$$[L_1] + [L_2] = [X] + [L_1 \times_X L_2] = [X] = [L_1] + [DL_1] = [L_1] + [L_1]$$

so that the claim follows. □

We note that the exact sequence

$$L_1(\mathcal{C}, ♀) \xrightarrow{\text{fgt}} K_0(\mathcal{C})_{C_2} \xrightarrow{\text{hyp}} \mathrm{GW}_0(\mathcal{C}, ♀) \to L_0(\mathcal{C}, ♀) \to 0$$

will later be continued to the left by the higher Grothendieck–Witt groups $\mathrm{GW}_i(\mathcal{C}, ♀)$ and the left terms are the higher homotopy groups of the spectrum $K(\mathcal{C})_{hC_2}$.

Proposition 6.14. *For a ring R there are group homomorphisms*

$$\mathrm{GW}^s_0(R) \to \mathrm{GW}_0(\mathcal{D}^{\mathrm{perf}}(R), ♀^s_R) \qquad \mathrm{GW}^q_0(R) \to \mathrm{GW}_0(\mathcal{D}^{\mathrm{perf}}(R), ♀^q_R)$$

where $\mathrm{GW}^s_0(R)$ and $\mathrm{GW}^q_0(R)$ are defined as the respective group completions.

We will also see in the next section using algebraic surgery that the second homomorphism here is an isomorphism. For the first one to become an isomorphism we have to replace the quadratic functor $♀^s_R$ by a better one (the aforementioned non-abelian derived functor).

7. Algebraic surgery

In this section we want to explain the process of algebraic surgery, which allows to simplify Poincaré objects. More precisely we will show that any Poincaré object (X, q) for $(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^q_R)$ can be modified to a cobordant object (X', q'), i.e. they represent the same class in $L^q_0(R)$, such that $X' = P[0]$ is concentrated in degree 0. This will be the key step to compute the group $L^q_0(R)$.

As a first step we recall the notion of algebraic bordisms. So far we have only defined null-bordisms. Throughout we work in a fixed Poincaré-∞-category $(\mathcal{C}, \Qoppa)$.

Definition 7.1. *An algebraic bordism between Poincaré objects (X, q) and (X', q') is given by a Lagrangian in $(X \oplus X', q \oplus -q')$.*

As before we think of those as spans

$$\begin{tikzcd} & L \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X & & Y \end{tikzcd}$$

together with a path between the two forms $q_X \mid_L$ and $q_Y \mid_L$ such that a certain non-degeneracy condition is satisfied: a priori it means that the sequence

$$L \xrightarrow{(i,j)} X \oplus Y \cong DX \oplus DY \xrightarrow{(i,-j)} DL$$

is a fibre sequence (the sign comes from the fact that we have taken the opposite form in Y, it is really in the identification of Y with DY but we have placed in the map to take the standard identification. But this is of course equivalent to the assertion that the square

$$\begin{tikzcd} & L \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[dr, "Di"'] & & Y \arrow[dl, "Dj"] \\ & DL & \end{tikzcd}$$

with the induced filler is a pullback.

Definition 7.2. *A surgery datum in $(\mathcal{C}, \Qoppa)$ is given by a Poincaré object (X, q) together with a map $s : S \to X$ and a specified nullhomotopy of $q|_S$.*

Remark 7.3. This definition is of course highly inspired by geometric surgery. In fact a geometric surgery datum on a closed n-manifold M, that is an embedding $i : S^p \times D^{n-p} \to M$ leads to an algebraic surgery datum on the associated cochains $C^*(M, \mathbb{Z})$ using the induced map

$$S = \mathbb{Z}[p - n] \to C^*(M, \mathbb{Z})$$

under Poincaré duality. This is just the cohomology class in $H^{n-p}(M)$ Poincaré dual to the homology class $[i] \in H_p(M)$. The nullhomotopy of the restricted form can then be obtained as follows. We consider the trace of the surgery

$$W = (M \times [0, 1]) \coprod_{S^p \times D^{n-p} \times \{1\}} \left(D^{p+1} \times D^{n-p}\right) \simeq M \coprod_{S^p} D^{p+1}$$

and the result of surgery

$$M' = \Big(M \setminus (S^p \times \mathring{D}^{n-p})\Big) \coprod_{S^p \times S^{n-p-1}} \big(D^{p+1} \times S^{n-p-1}\big)$$

Then one has that W is a cobordism between M and M'. Consequently we have that the images of the fundamental classes of M and M' in W agree. As a result, the restrictions of the fundamental symmetric Poincaré forms on $C^*(M)$ and $C^*(M')$ to $C^*(W)$ agree. Now we have the following two facts:

(1) There is a factorization

$$S \to C^*(W) \to C^*(M),$$

i.e. the class in $H^{n-p}(M)$ comes pulled back from $H^{n-p}(W)$ (this is true since the pushforward of $[i] \in H_p(M)$ to $H_p(W)$ vanishes (by construction).

(2) The composite $S \to C^*(W) \to C^*(M')$ vanishes, this is in fact a fibre sequence.

Together these facts imply that the form on $C^*(M)$ pulled back to S vanishes and provides us with a canonical nullhomotopy. Moreover we see that we can assemble everything into a diagram

$$\begin{tikzcd} & C^*(W) \arrow[dl] \arrow[dr] & \\ C^*(M') & & C^*(M) \end{tikzcd}$$

where S is the fibre of the map $C^*(W) \to C^*(M')$ and the fibre of the right map is induced by the dual class $S^{n-p-1} \to M'$.

Clearly every Lagrangian gives rise to a surgery datum but we do not require S to be Lagrangian here. In fact, the process that we are going to describe will take advantage of the failure to be Lagrangian. To this extend note that for a surgery datum we get a specified nullhomotopy of the composite

$$S \xrightarrow{x} X = DX \xrightarrow{Dx} DS\ .$$

One can think of this as a sort of '2-term chain complex' in the stable ∞-category $\mathcal{C}$ and what we want to do not is to measure its failure to be a fibre sequence.

Lemma 7.4. *For a given sequence $A \to X \to B$ in a stable ∞-category with a specified nullhomotopy we have a canonical equivalence*

$$\mathrm{cof}(A \to \mathrm{fib}(X \to B)) \simeq \mathrm{fib}(\mathrm{cof}(A \to X) \to B)\ .$$

We will denote this object by $\mathcal{H}(A \to X \to B)$ where the $\mathcal{H}$ is for 'homology'.

Proof. We consider the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

as a morphism from $A \to X$ to $0 \to B$ in the arrow category $\mathcal{C}^{\Delta^1}$. Then the functor

$$\mathrm{cof} : \mathcal{C}^{\Delta^1} \to \mathcal{C}$$

is exact, so it in particular preserves fibres. Thus first taking the fibre in the arrow category and then the cofibre is the same as first taking the cofibre functor and then the fibre. This shows the claim. □

Remark 7.5. The homology $\mathcal{H}(A \to X \to B)$ admits a two step 'filtration'

$$\mathcal{H}(0 \to 0 \to B) \to \mathcal{H}(0 \to X \to B) \to \mathcal{H}(A \to X \to B)$$

which is

$$B[-1] \to \mathrm{fib}(X \to B) \to \mathcal{H}(A \to X \to B) \ .$$

The 'associated graded' of this filtration recovers the initial 'chain complex' since taking homology is an exact process. This is the first step in an equivalence between filtered objects and chain complexes in $\mathcal{C}$.

In the situation of a surgery datum $S \xrightarrow{s} X$ (we abusively as before suppress the forms and nullhomotopies in the notation) we write

$$X_s := \mathcal{H}(S \xrightarrow{x} X \xrightarrow{Dx} DS) \qquad \text{and} \qquad L_s := \mathrm{fib}(X \to DS) \ .$$

There are by definition maps $L_s \to X_s$ (with fibre S) and $L_s \to X$ (with cofibre DS).

One should think of X_s as been obtained by a 2-step process: first killing $S \to X$, i.e. taking the cofibre X/S. But then the result does not have Poincaré duality anymore, since $D(X/S) = \mathrm{fib}(X \to DS) = L_s$ which is not equivalent to X/S unless $S = 0$. This is then remedied by taking the fibre of $X/S \to DS$ as we then get

$$D(X_s) = D(\mathcal{H}(S \to X \to DS)) = X_s$$

by the self duality in the definition of homology. In fact we get a pullback square

$$\begin{tikzcd} & L_s \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X_s \arrow[dr] & & X \arrow[dl, "Dj"] \\ & DL_s & \end{tikzcd}$$

which is 'self-dual'.

Proposition 7.6. *In the situation above, the object X_s has a canonical structure q_s of a Poincaré object and $X_s \leftarrow L_s \to X$ is canonically an algebraic bordism between (X, q) and (X_s, q_s) refining the above square. As in geometry we say that (X_s, q_s) is obtained from (X, q) by performing surgery along s and that L_s is the trace of the surgery.*

We first need a preparatory Lemma:

Lemma 7.7. *Let $(\mathcal{C}, \Omega)$ be a Poincaré-∞-category and $A \to B \to C$ be a fibre seuqence. Then there is an induced fibre sequence*

$$\Omega(C) \to \Omega(B) \to \Omega(A) \times_{B_\Omega(A,A)} B_\Omega(A, B).$$

Proof. We note that we have a commutative diagram

$$\begin{tikzcd}
\Qoppa(C) \arrow[r] \arrow[d] & \Qoppa(B) \arrow[r] \arrow[d] & \Qoppa(A) \arrow[d] \\
B_\Qoppa(C,C) \arrow[r] & B_\Qoppa(B,B) \arrow[r] & B_\Qoppa(A,A)
\end{tikzcd}$$

factoring the lower morphisms through $B_\Qoppa(A,B)$ we get a square

$$\begin{tikzcd}
\Qoppa(C) \arrow[r] \arrow[d] & \Qoppa(B) \arrow[r] \arrow[d] & \Qoppa(A) \arrow[d] \\
0 \arrow[r] & B_\Qoppa(A,B) \arrow[r] & B_\Qoppa(A,A)
\end{tikzcd}$$

with compatible horizontal nullhomotopies. The right hand square gives rise to the morphism $\Qoppa(B) \to \Qoppa(A) \times_{B_\Qoppa(A,A)} B_\Qoppa(A,B)$ and the nullhomotopies of the horizontal maps induce a map from $\Qoppa(C)$ to the total fibre of the right hand square, i.e. the sequence

$$\Qoppa(C) \to \Qoppa(B) \to \Qoppa(A) \times_{B_\Qoppa(A,A)} B_\Qoppa(A,B)$$

in question with its nullhomotopy. Now we need to specify that it is a fibre sequence. The whole sequence is clearly exact and functorial in $\Qoppa$. Thus by the structure theory for quadratic functors it thus suffices to show this claim for $\Qoppa$ linear and $\Qoppa$ homogenous. The first case is obvious and for the second we note that we get a fibre sequence

$$B_\Qoppa(C,C) \to B_\Qoppa(B,B) \to B_\Qoppa(B,A) \times_{B_\Qoppa(A,A)} B_\Qoppa(A,B)$$

using bilinearity. Thus applying $(-)_{hC_2}$ implies the claim since

$$B_\Qoppa(B,A) \times_{B_\Qoppa(A,A)} B_\Qoppa(A,B) = B_\Qoppa(A,A) \times_{B_\Qoppa(A,A) \oplus B_\Qoppa(A,A)} B_\Qoppa(A,B) \oplus B_\Qoppa(B,A) \ .$$

which is straighforward to verify (it is the usual way of rewriting a pullback as an equalizer). $\square$

Remark 7.8. One can also use the fact that $\Qoppa$ sends pushouts to totalizations to deduce the claim: applying this we get that $\Qoppa(C)$ is equivalent to the limit of the diagram

$$\Qoppa(B) \rightrightarrows \Qoppa(A \oplus B) \Rrightarrow \Qoppa(A \oplus A \oplus B)$$

indexed over $\Delta_{\leq 2}$. It is not hard to deduce the formula also from this invoking the definition of the bilinear part. Similarly one can easily deduce it from the 2-excessiveness of $\Qoppa$ in Goodwillie's sense.

Proof of Proposition 7.6. We employ the square

$$\begin{tikzcd}
& L_s \arrow[dl, "i"'] \arrow[dr, "j"] & \\
X_s \arrow[dr] & & X \arrow[dl, "Dj"] \\
& DL_s = X/S &
\end{tikzcd}$$

with induced fibre sequence

$$S \to L_s \to X_s \ .$$

We get an induced fibre sequence

$$\Qoppa(X_s) \to \Qoppa(L_s) \to \Qoppa(S) \times_{\mathrm{map}(S,DS)} \mathrm{map}(L_s, DS)$$

We want to define an element q_s in $\Qoppa(X_s)$, that restricts to the pullback $q|_{L_s}$, so we have to give a nullhomotopy of $q|_{L_s}$ in $\Qoppa(S) \times_{\mathrm{map}(S,DS)} \mathrm{map}(L_s, DS)$. By assumption we have a specified nullhomotopy of the restriction of q to S and the map $L_s \to DS$ is canonically nullhomotopic as L_s is the fibre of the induced map $X \xrightarrow{Di} DS$. These two homotopies fit together by construction. Now one checks that the adjunct map of this form induces an equivalence $D(X_s) \simeq X_s$ and that the square above is induced by the structures that we have just constructed. □

Now we want to see how to use this abstract construction to simplify objects up to surgery. Let us therefore consider the situation of the Poincaré-∞-category $(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^q)$ and assume that we are given a Poincare object (X, q). By perfectness, we see that X will be bounded below but might have negative homology. Pick a negative homology class of lowest degree represented by a map

$$s : R[-k] \to X \ .$$

We have that $\Qoppa^q(R[-k]) = \mathrm{map}(R[-2k], R)_{hC_2} = R[2k]_{hC_2}$ is $2k$-connective. In particular it has vanishing π_0 so that the restriction $q|_{R[-k]}$ vanishes (by a unique homotopy as also π_1 vanishes). We thus can perform surgery along s and obtain an object

$$X_s = \mathcal{H}\left(R[-k] \to X \to R[k]\right)$$

The cofibre $X/R[-k]$ has as lowest homology $H_{-k}(X)/s$ and this lowest homology agrees with the lowest homology of X_s. As a result we see that we can inductively get rid of the lowest homology. This proves the following:

> Every Poincaré object in $(\mathcal{D}^{\mathrm{perf}}(R), \Qoppa^q)$ is bordant to one which is connective.

Now let us analyze how such connective Poincaré objects X look like: first of all, we get that $DX \simeq X$ is also connective. Thus it follows that $X = D(DX) = \mathrm{RHom}(DX, R)$ is coconnective, that it has vanishing homology above degree 0, thus X is given by $P[0]$ for an R-module P. But in fact more is true.

Lemma 7.9. *Assume that for a perfect chain comples X of R-modules we have that X is a-connective and the dual DX (as an R^{op}-module) is $(-b)$-connective. Then X has Tor-Amplitude in $[a, b]$ and consequently can be represented by a finite projective chain complex*

$$\ldots \to 0 \to P_b \to P_{b-1} \to \ldots \to P_a \to 0 \to \ldots$$

supported in the interval $[a, b]$ (see Remark 2.11).

Proof. Clearly X has Tor-Amplitude $\geq a$. We thus have to show that

$$X \otimes_R N[0]$$

is b-truncated for any right R-module N. But we have

$$X \otimes_R N[0] \simeq \mathrm{map}_R(DX, N[0])$$

and the latter space is b-truncated as we have that

$$\tau_{\geq b+1}(\mathrm{map}_R(DX, N[0])) = \tau_{\geq 0}(\mathrm{map}_R(DX[b+1], N[0])) = \mathrm{Map}_R(DX[b+1], N[0])$$

which vanishes since $DX[b+1]$ is 1-connective. □

In our situation this claim amounts to the assertion that

> Every Poincaré object in $(\mathcal{D}^{\mathrm{perf}}(R), Ϙ^q)$ is bordant to a Poincaré object $(P[0], q)$ for P finitely generated projective, i.e. one given by a classical quadratic unimodular form.

In particular we see that the map $W_0^q(R) \to L_0^q(R)$ is surjective. We actually want to show that this map is an isomorphism. We thus have to do more: we have to show that for a classical quadratic form (P, q) a Lagrangian $L \to P[0]$ exists precisely if q is metabolic, that is a Lagrangian concentrated in degree 0, i.e. of the form $L'[0]$ such that the sequence $L' \to P \to DL'$ is short exact, in particular $L' \to P$ is injective. However, if a Lagrangian $L \to P[0]$ happens to already be connective then it follows that DL is connective as it is the cofibre of $L \to P[0]$ and thus L is also coconnective, thus concentrated in degree 0. Moreover it also follows that DL is concentrated in degree 0. Thus the sequence in question is automatically short exact and all modules are finitely generated projective.

Thus it suffices for our purposes to show that we can modify a given Lagrangian so that we get a connective Lagrangian. We want to employ a relative version of surgery to simplify a Lagrangian. To this end consider the following setup for a general Poincaré-∞-category:

Given a Lagrangian $L \to X$ in a Poincaré object X and assume we have a map $s : S \to L$ with a nullhomotopy $S \to L \to X$. Then we get a form of degree 1 on S by composing the two nullhomotopies of the restriction of q to S (one coming from the Lagrangian structure and one coming from the nullhomotopy of the composition $S \to L \to X$). Now assume that we have a nullhomotopy of this degree 1 form, or equivalently a homotopy between the two path from $q|_S$ to zero. We can equivalently interpret this datum as a surgery datum of the form

$$(S \to 0) \to (L \to X)$$

in $\mathrm{Met}(\mathcal{C}, Ϙ)$. Thus we can perform algebraic surgery in $\mathrm{Met}(\mathcal{C}, Ϙ)$ and get a new object, which is of the form

$$\mathcal{H}\left(S \to L \to D(S)[-1]\right) \to X.$$

Now let us apply this process to our situation: we have a Lagrangian

$$L \to P[0]$$

in $(\mathcal{D}^{\mathrm{perf}}(R), Ϙ^q)$ with P finitely generated projective. Then we again pick a homology class $S = R[-k] \to L$ of lowest degree. We find that

$$Ϙ^q(R[-k]) = \mathrm{map}(R[-2k], R)_{hC_2} = R[2k]_{hC_2}$$

is at least 2-connective so that we can choose all the necessary data and perform surgery. But then L/S has lowest homology given by

$$H_{-k}(L/S) = H_{-k}(L)/s\ .$$

From the fibre sequence

$$\mathcal{H} \to L/S \to D(0/S) = R[k-1]$$

describing $\mathcal{H} = \mathcal{H}\left(S \to L \to D(S)[-1]\right)$ we see that for $k \geq 2$ the lowest degree homology of $\mathcal{H}$ and X/S agrees. But for $k = 1$ we get an exact sequence

$$... \to R \to H_{-1}(\mathcal{H}) \to H_{-1}(L)/s \to H_0(R[1]) = 0\ .$$

so that we can unforunately not get rid of the (-1) first homology.

Thus the result is that by surgery we can product a Lagrangian L that is (-1) connective, i.e. in $\tau_{\geq -1}\mathcal{D}^{\mathrm{perf}}(R)$. It then follows from the cofibre sequence

$$L \to P[0] \to DL$$

that DL is connective which then by Lemma 7.9 implies that L has Tor-amplitude in $[-1, 0]$, i.e. can be represented by a chain complex of the form

$$L = (... \to 0 \to L_0 \to L_{-1} \to 0 \to ...)\ .$$

In particular we have maps

$$L_{-1}[-1] \to L \to P[0]$$

We consider the morphism in $\mathrm{Met}(\mathcal{C}, \Omega)$ given by

$$\left(L_{-1}[-1] \xrightarrow{\mathrm{id}} L_{-1}[-1]\right) \to (L \to P[0]) \tag{2}$$

we have that $\Omega_{\mathrm{met}}(L_{-1}[-1]) = 0$ so that we can equip this map with the structure of a surgery datum The lower map $L_{-1}[-1] \to P[0]$ is zero since L_{-1} is projective so that the relevant Ext-group vanishes.

Now we want to perform surgery on this map. In order to identify the outcome we need the following general assertion:

Lemma 7.10. *Assume that we have an arbitrary Poincaré category $(\mathcal{C}, \Omega)$ with an object X. For any object S we consider the zero map $s : S \to X$ together with any nullhomotopy of the pulled back form $q\mid_S$.*[14] *Then performing surgery on s produces a direct sum*

$$X_s = X \oplus M$$

where M is metabolic with Lagrangian $D(S[1])$.

Proof. We can write the surgery datum $S \to X$ as the sum of surgery data $S \to 0$ and $0 \to X$ where the first carries the nullhomotopy $\gamma \in \Omega(S)$ assumed in the lemma. Then the surgery outcoming is clearly also a direct sum of the surgeries: thus $X_s = X \oplus M$ where M is obtained by surgery along $S \to 0$. But then there is a coboridsm between M and 0 as the trace of this surgery, thus a nullbordism given by $\mathrm{fib}(0 \to DS) = D(S[1])$. □

Applying this lemma we see that performing surgery on the map (2) in $\mathrm{Met}(\mathcal{C}, \Omega)$ the result takes the form

$$L_0 \to P[0] \oplus M$$

where M has $L_{-1}[0]$ as a Lagrangian and is concentrated in degree 0. Thus in total we arrive at the following conclusion.

> If a Poincaré object $P[0]$ in $(\mathcal{D}^{\mathrm{perf}}(R), \Omega^q_R)$ admits a Lagrangian nullbordism, then there is a classical metabolic M such that $P \oplus M$ is also metabolic in the classical sense.

As a result of this whole discussion we have the following result.

[14] There is a canonical nullhomotopy and in a previous version of this lemma we assumed that the nullhomotopy question was this canonical one. This leads to a slighly stronger conclusion (hyperbolic instead of metabolic) but will not be satisfied in all of our cases of interest.

Proposition 7.11. *For any ring R with involution the canonical homomorphism*

$$\mathrm{W}_0^q(R) \to L_0(\mathcal{D}^{\mathrm{perf}}(R), \Omega_R^q)$$

(see Proposition 6.8) considering a classical quadratic form as a Poincaré object is an isomorphisms.[15]

Proof. We have already seen surjectivity. For injectivity assume that a class $[P]$ lies in the kernel. Then we know that $P[0]$ admits a Lagrangian and thus by our conclusion above that $P \oplus M$ is metabolic. Thus in the Witt group $\mathrm{W}_0^q(R)$ we have

$$[P] = [P] + [M] = [P \oplus M] = 0$$

which shows the claim. □

Let us revisit quickly what we needed for the whole argument: we have used that the quadratic functor Ω_R^q has the following properties:

(1) We have $\pi_0(\Omega_R^q(R[-k])) = 0$ for $k > 0$ so that we can perform surgery.
(2) The dual DR is connective so that in the sequence

$$R[-k] \to X \to (DR)[k]$$

we do not get contributions messing up the surgery process (the critical case is $k = 1$).
(3) For connective X we have that $D(X)$ has Tor-Amplitude ≤ 0.
(4) The group

$$\pi_0 \Omega_{\mathrm{met}}(R[-k] \to 0) = \pi_1 \Omega(R[-k])$$

vanishes for $k \geq 2$.
(5) For the last step we again needed DL_{-1} to be connective so that we get that the metabolic M and its Lagrangian are connective.

Proposition 7.12. *These assumptions (1)-(4) are satisfied for an arbitrary Poincaré structure $(\mathcal{D}^{\mathrm{perf}}(R), \Omega)$ on $\mathcal{D}^{\mathrm{perf}}(R)$ precisely if*

- *The duality D preserves the full subcategory $\mathrm{Proj}_R \subseteq \mathcal{D}^{\mathrm{perf}}(R)$ of finitely generated projective modules (or equivalently $D(R)$ is finitely generated projective in degree 0).*
- *The value $\Omega(R)$ is connective.*

Note that under the first assumption, the second is equivalent to the assertion that $L_\Omega(R)$ is connective using the fibre sequence

$$\mathrm{map}_R(R, DR)_{hC_2} \to \Omega(R) \to L_\Omega(R).$$

as the first term is connective.

Proof. We first show that our conditions are necessary: condition (2) above implies that for a projective module $P[0]$ we have that $D(P[0])$ is connective by passing the direct sums and summands. But condition (3) then implies that it has Tor-Amplitude in $[0, 0]$, i.e. is finitely generated projective.

For the connectivity of $\Omega(R[-k])$ we note that $\Omega(R[-1])$ sits in a sequence

$$(\mathrm{map}_R(R, DR)[2k])_{hC_2} \to \Omega(R[-k]) \to L_\Omega(R)[k]$$

[15] We recall once more that the source group is precisely defined as in the symmetric case, just replacing symmetric forms by quadratic forms and Lagrangians.

so that condition (1) implies that

$$\pi_{-k}(L_{\Omega}(R)) = \pi_0(L_{\Omega}(R)[-k]) = \pi_0(\Omega(R[-k])) = 0 \tag{3}$$

for all $k \geq 0$ so that we get the desired connectivity.

Now we need to argue that our conditions are in fact sufficient. To this end assume that Ω satisfies the assumptions of the Proposition. Then condition (1) above is satisfied by reversing equation (3). Condition (2) is clear. For condition (3) we write X as a finite iterated colimit of finitely generate projective modules concentrated in degree zero. Then DX is a finite iterated limit of finitely generated projective modules, thus has Tor-Amplitude ≤ 0. For condition (4) we again observe that by the fibre sequence

$$(\mathrm{map}_R(R, DR)[2k])_{hC_2} \to \Omega(R[-k]) \to L_{\Omega}(R)[k]$$

we have that π_1 vanishes for $k \geq 1$. The last condition is automatically satisfied now. □

Definition 7.13. *We say that a quadratic functor Ω on $\mathcal{D}^{\mathrm{perf}}(R)$ is* compatible with the weight structure *if it satisfies the assumptions of Proposition 7.12.*

Example 7.14. The functor Ω^q_R is compatible with the weight structure. The functor Ω^s_R is not since

$$\Omega^s_R(R) = R^{hC_2}$$

is not connective.

Definition 7.15. *In the situation of a quadratic functor Ω compatible with the weight structure we define a group $W_0^{\Omega}(R)$ as follows: it is generated by equivalence classes of Poincaré objects whose underlying object is of the form $P[0]$ (we shall refer to this as strictly 0-dimensional) and we take the quotient by objects that admit a strictly 0-dimensional Lagrangian $L[0] \to P[0]$.*

Theorem 7.16. *For every Ω which is* compatible with the weight structure *the canonical morphism $W_0^{\Omega}(R) \to L_0(\mathcal{D}^{\mathrm{perf}}(R), \Omega)$ is an isomorphism.*

Remark 7.17. We note that the definition of $W_0^{\Omega}(R)$ can actually be simplified: we claim that it only depends on the functor

$$\pi_0\Omega : \mathrm{Proj}_R^{\mathrm{op}} \to \mathrm{Ab}\ .$$

To see this we note that up to isomorphism we only need to fix the class $q \in \pi_0(\Omega(X))$ to say that an object is Poincaré, since from this we can still recover the homotopy class $X \to DX$ as the image under the map

$$\pi_0(\Omega(X)) \to \pi_0(B_{\Omega}(X, X)) = [X, DX]\ .$$

The crucial point is though, that for a given Poincaré form $q \in \Omega(P[0])$ with a nullhomotopy h of the restricted form $q|_L$ for some projective $L[0]$ to sequence

$$L[0] \to P[0] \to DL[0]$$

is a fibre sequence precisely if it is a short exact sequence of projective modules and this is independent of the choice of h! Therefore we can neglect the choice of h.

Let us see what this means for the other quadratic L-groups. We have already seen that

$$(\mathcal{D}^{\mathrm{perf}}(R), \Omega^q[2n]) \simeq (\mathcal{D}^{\mathrm{perf}}(R), \Omega^q_{(-1)^n})$$

where the right hand side was the $(-1)^n$-quadratic functor. This is also compatible with the weight structure so that we get the following result.

Corollary 7.18. *For any ring with involution R we have isomorphisms*

$$W_0^{q,(-1)^n}(R) \xrightarrow{\simeq} L_{2n}^q(R) \qquad (P,q) \mapsto (P[-n],q)$$

for any n where the source is the $\pm$-quadratic Witt group of R.

Now we would like to understand the odd L-group and in general the higher L-groups of a general Poincaré structure $Ϙ : \mathcal{D}^{\mathrm{perf}}(R)^{\mathrm{op}} \to \mathrm{Sp}$ that is compatible with the weight structure. In this situation we for for every n an equivalence

$$(\mathcal{D}^{\mathrm{perf}}(R), Ϙ[-2n]) \xrightarrow{[n]} (\mathcal{D}^{\mathrm{perf}}(R), Ϙ')$$

where $Ϙ'(X) = Ϙ(X[n])[-2n]$ which has the same duality $X \mapsto D_Ϙ X$. In the quadratic situation the new functor $Ϙ'$ is also compatible with the weigh structure. In general this does not happen. Indeed, we find that $L_{Ϙ'}(R) = L_Ϙ(R)[-n]$. This being connective for every R then forces $L_Ϙ(R) = 0$ which is equivalent to the assertion that $Ϙ = Ϙ_B^q$ is quadratic (on a given symmetric bilinear part B).

Let us thus analyse the surgery process for Poincaré objects X of dimension n (not necessarily even) for a given quadratic functor $Ϙ$. We pick a class in least negative degree $s : R[-k] \to X$ as before. Now in order to be able to perform surgery we need that $\pi_n(Ϙ(R[-k]) = 0$. This is satisfied as long as $k > n$ by the assumptions on $Ϙ$. Thus we can perform surgery. The outcome X_s of surgery will sit in a fibre sequence

$$X_s \to X/s \to D(R[-k])[-n] = D(R)[k-n]$$

In order for H_{-k} to be unaffected by the long exact sequence

$$H_{-k+1}(D(R)[k-n]) \to H_{-k}(X_s) \to H_{-k}(X)/s \to H_{-k}(D(R)[k-n])$$

we thus need that $k - n > -k$ or equivalently $2k > n$. If $n \geq 0$ then the second condition is automatic if $k > n$ so that we can perform surgery to improve X to become $-n$-connective. Then we have

$$DX = (DX[-n])[n] = X[n]$$

will be connective so that we conclude that X has Tor-Amplitude in $[-n,0]$ and arrive at the conclusion

> Every Poincaré object of dimension $n \geq 0$ is cobordant to one with Tor-Amplitude in $[-n,0]$.

Definition 7.19. *A strictly n-dimensional Poincaré object $X \in \mathcal{D}^{\mathrm{perf}}(R)$ is an n-dimensional Poincaré object with Tor-Amplitude in $[-n,0]$. Such an object is called strictly metabolic if it admits a Lagrangian $L \to X$ which also has Tor-Amplitude in degrees $[-n,0]$. We define for any Poincaré structure the classical L-groups as*

$$\mathrm{L}_{\mathrm{cl},n}^{Ϙ}(R) = \frac{\{\textit{strictly } n\textit{-dimensional Poincaré objects over } R\}}{\{\textit{strictly metabolic Poincaré objects}\}} \;.$$

Theorem 7.20. *For $n \geq 0$ and $Ϙ$ compatible with the weight structure the canonical map*

$$\mathrm{L}_{\mathrm{cl},n}^{Ϙ}(R) \to L(\mathcal{D}^{\mathrm{perf}}(R), Ϙ)$$

is an isomorphism.

Proof. A priori we again have to be careful with the equivalence relation since by surgery on a Lagrangian $L \to X$ we again can only bring it in degrees $\geq -n-1$. But then we perform a similar trick as above to get rid of a bottom cell $L_{-n-1}[-n-1] \to L$ to the price of replacing X by $X \oplus M$ with M strictly metabolic. $\square$

What happens for negative n? The same analysis applies so that to kill a class in degree $-k$ we need that $k > n$ and that $2k > n$ or equivalently $-k < -n$ and $-k < -n/2$. We conclude that we can always achieve that the homology groups of X vanish below degrees $-n/2$. Then we find that

$$DX = (DX[-n])[n] = X[n]$$

has homology vanishing below degrees $n/2$ so that we arrive at the following conclusion

Every Poincaré object of dimension $n \leq 0$ is cobordant to one with Tor-Amplitude in degrees $[\lfloor -n/2 \rfloor, \lceil -n/2 \rceil]$.

Thus it might be concentrated in either 1 or 2 adjacent degrees depending on whether n is even or odd. Note that for the quadratic functor a similar analysis also applies in positive degrees and thus there we also have this conclusion for positive n.

Definition 7.21. *We say that a Poincaré object $X \in \mathcal{D}^{\mathrm{perf}}(R)$ of dimension n is concentrated in the middle dimension if it has Tor-Amplitude in $[\lfloor -n/2 \rfloor, \lceil -n/2 \rceil]$. We say that such an object admits a middle dimensional Lagrangian if it admits a Lagrangian with Tor-Amplitude in $[\lfloor -n/2 \rfloor, \lceil -n/2 \rceil]$. We define the middle dimensional L-groups aka. (higher and lower) Witt groups as*

$$W_n^{\Omega}(R) := \frac{\{n\text{-dimensional Poincaré objects concentrated in the middle dimension}\}}{\{\text{those that admit a Lagrangian in the middle dimension}\}}$$

Proposition 7.22. *For a Poincaré stucture $\Omega : \mathcal{D}^{\mathrm{perf}}(R)^{\mathrm{op}} \to \mathrm{Sp}$ compatible with the weight structure we have that the canonical morphism*

$$W_n^{\Omega}(R) \to L_n(\mathcal{D}^{\mathrm{perf}}(R), \Omega) \ .$$

is an isomorphisms for $n \leq 0$. For Ω^q this is true in all degrees.

Proof. Again we only have to argue that we can get rid of the additional problems that arise with Lagrangians which works precisely as before. $\square$

Remark 7.23. We will see later that this is also true for other quadratic functors under additional assumptions (e.g. R a Dedekind ring).

One can also give a rather explicit algebraic description of these Poincaré objects in the middle dimension for n odd in terms of so-called *formations*. This leads to a completely algebraic description of all the Witt-groups which is important to understand them explicitly as we will see soon.

8. Weight structures

Definition 8.1. *A weight structure on a stable ∞-category $\mathcal{C}$ consists of a pair of full, saturated subcategories*

$$\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0} \subseteq \mathcal{C}$$

such that the following are satisfied:

(1) *$\mathcal{C}_{w\geq 0}$ is closed under coproduct, retracts and positive shifts and $\mathcal{C}_{w\leq 0}$ is closed under coproducts, retracts and negative shifts.*

(2) *For* $X \in \mathcal{C}_{w\leq 0}$ *and* $Y \in \mathcal{C}_{w\geq 1} := \mathcal{C}_{w\geq 0}[1]$ *we have that* $[X,Y] = 0$.
(3) *For any* $X \in \mathcal{C}$ *there is a cofibre sequence*

$$X' \to X \to X''$$

with $X' \in \mathcal{C}_{w\leq 0}$ *and* $X'' \in \mathcal{C}_{w\geq 1}$.

For a general weight structure we shall write

$$\begin{aligned}
&\mathcal{C}_{w\geq n} := \mathcal{C}_{w\geq 0}[n]\\
&\mathcal{C}_{w\leq n} := \mathcal{C}_{w\leq 0}[n]\\
&\mathcal{C}_{w\in[a,b]} := \mathcal{C}_{a\leq w\leq b} := \mathcal{C}_{w\geq a} \cap \mathcal{C}_{w\leq b}\\
&\mathcal{C}^{w\heartsuit} := \mathcal{C}_{w\in[0,0]}\ .
\end{aligned}$$

The latter full subcategory will be refered to as the *weighty heart of* $\mathcal{C}$. The weight structure will be called *bounded* if every object $X \in \mathcal{C}$ lies in $\mathcal{C}_{w\in[a,b]}$ for finite a and b.

Example 8.2. We consider the following weight structure on $\mathcal{D}^{\mathrm{perf}}(R)$: the subcategory $\mathcal{D}^{\mathrm{perf}}(R)_{w\geq 0}$ is given by the connective objects i.e. homology concentrated in non-negative degrees or equivalently Tor-amplitude ≥ 0. The subcategory $\mathcal{D}^{\mathrm{perf}}(R)_{w\leq 0}$ consists of those objects of Tor-amplitude ≤ 0. In order to see that this defines a weight structure we first note that these two subcategories are clearly closed under finite colimits respectively limits. For the third condition we simply note that for a given perfect complex

$$P = (... \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to ...)$$

we can form new chain complexes

$$\begin{aligned}
P' &= (... \to 0 \to 0 \to P_0 \to P_{-1} \to ...)\\
P'' &= (... \to P_2 \to P_1 \to 0 \to ...)
\end{aligned}$$

so that we get a cofibre sequence

$$P' \to P \to P''$$

and clearly $P' \in \mathcal{D}^{\mathrm{perf}}(R)_{w\leq 0}$ and $P'' \in \mathcal{D}^{\mathrm{perf}}(R)_{w\geq 1}$. Finally for a bounded projective complex in negative degree

$$P = (... \to 0 \to 0 \to P_0 \to P_{-1} \to ...)$$

maps into any chain complex C concentrated in degrees ≥ 1 are clearly zero (we do not need to resolve or anything since P is already projective and bounded below). The heart consists of the full subcategory $\mathrm{Proj}_{\mathbb{Z}} \subseteq \mathcal{D}(\mathbb{Z})$, given by perfect complexes with Tor-Amplitude in $[0,0]$ which happens to be an ordinary category (considered as an ∞-category).

Lemma 8.3. *We have equalities*

$$\begin{aligned}
\mathcal{C}_{w\geq 0} &= \{B \in \mathcal{C} \mid [A,B] = 0 \textit{ for all } A \in \mathcal{C}_{w\leq -1}\}\\
\mathcal{C}_{w\leq 0} &= \{A \in \mathcal{C} \mid [A,B] = 0 \textit{ for all } B \in \mathcal{C}_{w\geq 1}\}
\end{aligned}$$

Both of these subcategories are closed under extensions.

This in particular shows that a weight structure is overdetermined and can be uniquely described by any of the full subcategory $\mathcal{C}_{w\geq 0}$ or $\mathcal{C}_{w\leq 0}$ (similar to the case of t-structures that we will treat later).

Proof. We show the first equality the second follows dually. The inclusion $\subseteq$ is clear. Thus assume that we have B in the right hand set of objects. Then we choose $B' \to B \to B''$ with $B' \in \mathcal{C}_{w\leq -1}$ and $B'' \in \mathcal{C}_{w\geq 0}$. By assumption the map $B' \to B$ is zero so that $B'' = B \oplus B'[1]$ admits B as a retract. Thus $B \in \mathcal{C}_{w\geq 0}$.

For the closure under extensions note that for an extension $B' \to B \to B''$ with $B', B'' \in \mathcal{C}_{w\geq 0}$ we consider for any $A \in \mathcal{C}_{w\leq -1}$ the sequence

$$[A, B'] \to [A, B] \to [A, B'']$$

which is exact in the middle. Thus $[A, B] = 0$. □

Example 8.4. Let R be any ring. Consider $\mathcal{D}(R)$, the unbounded derived category. We set

$$\mathcal{D}(R)_{w\geq 0}$$

to be the connective chain complexes. Then we get from the previous Lemma that we have to set

$$\mathcal{D}(R)_{w\leq 0} = \{X \in \mathcal{D}(R) \mid [X, Y] = 0 \text{ for } Y \text{ 1-connective}\} .$$

Most of the axioms are clear. The main non-trivial thing to verify is to give the factorization for some X which we do as before: we represent X by a K-projective chain complex

$$X = (... \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to ...)$$

and construct the factorizatiom $X' \to X \to X''$ as

$$X' = (... \to 0 \to 0 \to P_0 \to P_{-1} \to ...)$$
$$X'' = (... \to P_2 \to P_1 \to 0 \to ...)$$

Now one easily sees (as before) that X' has non-positive weight and X has positive weight. In fact this shows that X has weight ≥ 0 or ≤ 0 or in $[a, b]$ preicsely if it can be represented by a K-projective chain complex in these exact degrees.

Remark 8.5. The condition that for a chain complex X there are no non-trivial maps to a connected chain complex is also called projective Amplitude ≤ 0. It is equivalent to the assertion that

$$\pi_i(\mathrm{map}(X, N[0]) = 0$$

for $i > 0$ and N an ordinary module. In general we can ask this to be true for i outside of any interval and then say that X has projective amplitude in this interval. This ends up being equivalent to be representable by a chin complex in these precise degrees. In the perfect case this is equivalent to having Tor amplitude in that intervall.

Example 8.6. We construct weight structures on $\mathrm{Sp}^{\mathrm{fin}}$ and Sp by letting the positive part be connective spectra. We then find that

$$\mathrm{Sp}_{w\leq 0} = \{X \in \mathrm{Sp} \mid X \text{ admits a cell structure with cells in degrees } \leq 0\}$$

and similar

$$\mathrm{Sp}^{\mathrm{fin}}_{w\leq 0} = \{X \in \mathrm{Sp} \mid X \text{ admits a cell structure with cells in degrees } \leq 0\}$$

The latter can equivalently be described as the set of all spectra X such that $H_*(X, \mathbb{Z})$ vanishes in positive degrees and is free in degree 0. The heart consists of the full subcategory $\mathrm{Proj}_{\mathbb{S}} \subseteq \mathrm{Sp}^{\mathrm{fin}}$ of all spectra that are of the form $\oplus\mathbb{S}$ (and similar for infinite sums in the non-finite case).

Lemma 8.7. *Any object $X \in \mathcal{C}_{w\in[a,b]}$ admits a finite cell structure, that is a filtration*

$$0 \to X_a \to X_{a+1} \to X_{a+2} \to \ldots \to X_b = X$$

with X_n/X_{n-1} in $\mathcal{C}_{w=n}$.

Proof. We proceed by induction over $b-a$. If $b=a$ then the claim is clear. In general let $X \in C_{w\in[a,b]}$ and chose a fibre sequence

$$X' \to X \to X''$$

with $X' \in \mathcal{C}_{w\leq b-1}$ and $X'' \in \mathcal{C}_{w\geq b}$. We claim that in fact $X' \in \mathcal{C}_{w\in[a,b]}$ and $X'' \in \mathcal{C}_{w=b}$. This then by inductively finishes the proof. To see the claim we first observe that X'' is an extension

$$X \to X'' \to X'[1]$$

where X and $X'[1]$ both have weight $\leq b$. Thus X'' also has weight $\leq b$ and thus weight exactly b. Similar we conclude from the extension

$$X''[-1] \to X' \to X$$

that X' has weight $\geq a$. $\square$

Remark 8.8. For a general object X there is an unbounded filtration

$$\ldots \to X_n \to X_{n+1} \to \ldots$$

with compatible maps $X_n \to X$ such that the weight of the cofibre $X_n \to X$ tends to ∞ with $n \to \infty$ and the weight of X_n tends to $-\infty$ with $n \to -\infty$. Under additional compatiblity assumptions (a non-degenerate weight structure compatible with sequential limits/colimits) this implies that $X = \varinjlim X_n$ and $\varprojlim X_n = 0$, i.e. that this is an complete exhaustive filtration.

Lemma 8.9. *The heart $\mathcal{C}^{w\heartsuit}$ is an additive ∞-category and any cofibre sequence $X' \to X \to X''$ in $\mathcal{C}$ with $X', X'' \in \mathcal{C}^{w\heartsuit}$ (and thus also $X \in \mathcal{C}^{w\heartsuit}$) splits, i.e. is equivalent to*

$$X' \xrightarrow{i_1} X' \oplus X'' \xrightarrow{p_2} X''$$

Proof. It clearly is additive as it is a full subcategory of a stable ∞-category (which is additive by Lemma 2.4). Thus it remains to show that every sequence splits. For this it suffices to show that the map $X'' \to X'[1]$ is zero. But this immediately follows since $X'[1]$ is in $C_{w\geq 1}$ and $X'' \in C_{w\leq 0}$. $\square$

Warning 8.10. *We do not claim that any cofibre sequence in the heart $\mathcal{C}^{w\heartsuit}$ itself splits. The seuqence in the heart that are cofibre sequences in $\mathcal{C}$ are precisely those sequences that are fibre and cofibre sequences in the heart.*

We recall that for a stable ∞-category $\mathcal{C}$ the K-theory $K_0(\mathcal{C})$ was defined as

$$K_0(\mathcal{C}) = \frac{\{\text{Isomorphism classes of objects in } \mathcal{C}\}}{[X] = [L \oplus X/L] \text{ for } L \to X \text{ a map}} .$$

For an additive ∞-category like $\mathcal{C}^{w\heartsuit}$ we define the K-theory simply as a the group completion of isomorphism classes of objects (under direct sum).

Proposition 8.11. *For a bounded weight structure on $\mathcal{C}$ the the canonical map*

$$K_0(\mathcal{C}^{w\heartsuit}) \to K_0(\mathcal{C}) \qquad P \mapsto P$$

is an isomorphism.

Proof. First note that this morphisms exists by the universal property of the group completion. First in order to see that this map is surjective we note that by Lemma 8.7 any object of $\mathcal{C}$ admits a finite filtration

$$0 \to X_a \to X_{a+1} \to X_{a+2} \to ... \to X_b = X$$

with X_n/X_{n-1} in $\mathcal{C}_{w=n}$. Thus in $K_0(\mathcal{C})$ we get that

$$[X] = \sum_n [X_n/X_{n-1}] = \sum_n (-1)^n [Y_n]$$

with $Y_n = X_n/X_{n-1}[-n] \in \mathcal{C}^{w\heartsuit}$. This shows surjectivity and also gives us a candidate for a potential inverse map:

$$[X] \mapsto \sum_n (-1)^n [Y_n]$$

for some 'cell' filtration on X. We want to argue that this is a well-defined map.

Assume first that X has weight in $[0, 1]$ and there are two maps

$$X_0 \to X \qquad \text{and} \qquad X'_0 \to X$$

from objects $X_0, X'_0 \in \mathcal{C}^{w\heartsuit}$ with respective cofibres X/X_0 and X/X'_0 in $\mathcal{C}_{w=1}$, i.e. $Y_1 = X/X_0[-1]$ and $Y'_1 = X/X'_0[-1]$ in the heart. In particular we have written X as X_0/Y_0 and X'_0/Y'_0. We will now show that in $K_0(\mathcal{C}^{w\heartsuit})$ the equality $[X_0] - [Y_1] = [X'_0] - [Y'_1]$ holds.

First we claim that there is a map $X_0 \to X'_0$ over X. To see this we simply lift the map $X_0 \to X$ through X'_0 uisng that the cofiber sits in $\mathcal{C}_{w\geq 1}$ and the orthogonality relation. Then we get a commutative square

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ Y'_1 \arrow[r] & X'_0 \end{tikzcd}$$

in which all terms lie in the heart. This is a pullback as the horizontal cofibres agree. Thus we have a fibre sequence

$$Y_1 \to X_0 \oplus Y'_1 \to X'_0$$

where all terms are in the heart. This splits by Lemma 8.9 and thus we get that $X_0 \oplus Y'_1 \simeq Y_1 \oplus X'_0$ which implies the equality in $K_0(\mathcal{C}^{w\heartsuit})$ that we want.

The general case work similar (by induction) but we do not want to introduce the necessary terminology for time reasons: one shows that for any two finite cell structures X_i and X'_i with shifted subquotients $Y_i, Y'_i \in \mathcal{C}^{w\heartsuit}$ on an object $X \in \mathcal{C}$ one gets an equivalence

$$\bigoplus_n \left(Y_{2n} \oplus Y'_{2n+1}\right) \simeq \bigoplus_n \left(Y_{2n+1} \oplus Y'_{2n}\right)$$

of objects in the heart. This then shows the necessary equality in $K_0(\mathcal{C}^{w\heartsuit})$. We recommend this in the case of $\mathcal{D}^{\mathrm{perf}}(R)$ as a nice exercise in homological algebra (hint: choose a map and pass to the mapping cone first). ☐

Definition 8.12. *A Poincaré structure* Ϙ *on* $\mathcal{C}$ *is compatible with a bounded weight structure if the following are satisfied:*

(1) *The duality restricts to a functor* $D : \left(\mathcal{C}^{w\heartsuit}\right)^{\mathrm{op}} \to \mathcal{C}^{w\heartsuit}$.

(2) *$\Qoppa$ sends $\mathcal{C}^{w\heartsuit}$ to connective spectra.*

We then speak of a weight structure on the Poincaré-∞-category $\mathcal{C}$.

Lemma 8.13. *Condition (1) is equivalent to:*

(1a) *The duality sends $\mathcal{C}_{w\leq 0}$ to $\mathcal{C}_{w\geq 0}$ and vice versa.*

Under condition (1) the second condition is equivalent to:

(2a) *$L_{\Qoppa}$ sends $\mathcal{C}^{w\heartsuit}$ to connective spectra.*
(2b) *$L_{\Qoppa}$ sends $\mathcal{C}_{w\leq 0}$ to connective spectra*
(2c) *$\Qoppa$ sends $\mathcal{C}_{w\leq 0}$ to connective spectra*

Proof. For $(1) \Rightarrow (1a)$ note that every object in note that every object in $\mathcal{C}_{w\leq 0}$ is an extension of objects in a single negative weight and for those it is clear that they are mapped to objects in a single positive degree. A similar argument works for the positive ones. The converse $(1a) \Rightarrow (1)$ is clear.

To see $(2) \Rightarrow (2a)$ use the fibre sequence

$$\mathrm{map}_{\mathcal{C}}(X, DX)_{hC_2} \to \Qoppa(X) \to L_{\Qoppa}(X) \tag{4}$$

and note that $\mathrm{map}_{\mathcal{C}}(X, DX)$ is connective as mapping spectra between objects in the heart are always connective. The implication $(2a) \Rightarrow (2b)$ works as $(1) \Rightarrow (1a)$ and for $(2b) \Rightarrow (2c)$ we simply use (4) again and the fact that $\mathrm{map}_{\mathcal{C}}(X, DX)_{hC_2}$ is connective by (1a). Finally $(2c) \Rightarrow (2)$ is clear. □

Note that this in particular shows that we can use the duality to characterize the objects of negative weight by the fact that the dual is of positive weight (as we have done in $\mathcal{D}^{\mathrm{perf}}(R)$).

Definition 8.14. *We say that a Poincaré object X of dimension n is strictly n-dimensional (with respect to a weight structure) if it lies in $\mathcal{C}_{w\in[-n,0]}$. Similary we say that it is concentrated in the middle dimension if it lies in $\mathcal{C}_{\lfloor -n/2\rfloor \leq w \leq \lceil -n/2\rceil}$. Similar for a Lagrangian we say that it is concentrated in the middle dimension if it is in $\mathcal{C}_{\lfloor -n/2\rfloor \leq w \leq \lfloor -n/2\rfloor}$.*

.

We define for a weight structure

$$\mathrm{L}_n(\mathcal{C}, \Qoppa, w) := \frac{\{\textit{strictly } n\textit{-dimensional Poincaré objects}\}}{\{\textit{strictly } n\textit{-dimensional metabolic Poincaré objects}\}}$$

for $n \geq 0$ and for $n \leq 1$ we define

$$\mathrm{L}_n(\mathcal{C}, \Qoppa, w) := \frac{\{\textit{middle dimensional } n\textit{-dimensional Poincaré objects}\}}{\{\textit{strictly } n\textit{-dimensional metabolic Poincaré objects}\}}$$

More generally we define the Witt group

$$W_n(\mathcal{C}, \Qoppa, w) = \frac{\{n\textit{-dimensional Poincaré objects concentrated in the middle dimension}\}}{\{\textit{those that admit a Lagrangian concentrated in the middle dimension}\}}$$

Theorem 8.15. *For a (not necessarily bounded) weight structure on a Poincaré ∞-category the canonical map*

$$\mathrm{L}_n(\mathcal{C}, \Qoppa, w) \to L_n(\mathcal{C}, \Qoppa)$$

is an isomorphism.

Proof. The proof works as before: for a given n-dimensional Poincaré object... □

One can in fact also generalize the argument we had in the case of $\mathcal{D}^{\mathrm{perf}}(R)$ to see that if $L_{\Qoppa}$ sends the heart to m-connective objects for some m then we can described the L-groups $L_n(\mathcal{C}, \Qoppa)$ by complexes conentrated in the middle dimension for $n \leq m$. In particular if the functor $\Qoppa$ is homogenous this works for all n. Also note that in particular the group $\mathrm{L}_0(\mathcal{C}, \Qoppa, w)$ only depends on the additive category $\mathcal{C}^{w\heartsuit}$ together with the restricted quadratic functor $\Qoppa|_{\mathcal{C}^{w\heartsuit}}$. In fact we can make the following definition:

Definition 8.16. *An additive Poincaré-∞-category is an additive ∞-category $\mathcal{A}$ together with a functor $\Qoppa : \mathcal{A}^{\mathrm{op}} \to \mathrm{Sp}_{\geq 0}$ such that we have a natural equivalence $B_{\Qoppa}(X, Y) \simeq \mathrm{map}_{\mathcal{A}}(X, DY)$ for some self equivalence $D : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$, where $B_{\Qoppa}$ is the second cross effect defined as in the stable situation.*

In such a situation we can define L_0 and it agrees with L_0 of the Poincaré-∞-category by the above result. We will in fact see soon that the additive Poincaré-∞-category $(\mathcal{C}^{w\heartsuit}, \Qoppa|_{\mathcal{C}^{w\heartsuit}})$ uniquely determines $(\mathcal{C}, \Qoppa, w)$.

We also want to deduce a similar results for the Grothendieck–Witt group. For an additive Poincaré-∞-category $(\mathcal{A}, \Qoppa)$ we define $\mathrm{GW}_0(\mathcal{A}, \Qoppa)$ as the group completion of isomorphism classes of Poincaré objects in $\mathcal{A}$.

Theorem 8.17. *For a given Poincare ∞-category with a bounded weight structure the canonical map*

$$\mathrm{GW}_0(\mathcal{C}^{w\heartsuit}, \Qoppa|_{\mathcal{C}^{w\heartsuit}}) \to \mathrm{GW}_0(\mathcal{C}, \Qoppa)$$

is an isomorphism.

Proof. We argue that the map is surective and injective. For surjectivity we note that by algebraic surgery we find that any Poincaré object X is cobordant to a Poincaré object X' concentrated in degree 0. Thus we find that in GW_0 that $X \oplus \overline{X'}$ admits a Lagrangian L, i.e. is equivalent to

$$[X] + [\overline{X'}] = [\mathrm{hyp}(L)]$$

for some $L \in \mathcal{C}$. Since $\overline{X'}$ lies in the image of our map it thus suffices to show that $[\mathrm{hyp}(L)]$ lies in the image. But this follows from the fact that the diagram

$$\begin{tikzcd}
K_0(\mathcal{C}^{w\heartsuit}) \arrow[r, "\mathrm{hyp}"] \arrow[d] & \mathrm{GW}_0(\mathcal{C}^{w\heartsuit}, \Qoppa|_{\mathcal{C}^{w\heartsuit}}) \arrow[d] \\
K_0(\mathcal{C}) \arrow[r, "\mathrm{hyp}"] & \mathrm{GW}(\mathcal{C}, \Qoppa)
\end{tikzcd}$$

commutes and the left vertical map is an isomorphism.

To be finished. □

Corollary 8.18. *We have an exact sequence*

$$L_1(\mathcal{C}, \Qoppa, w) \to K_0(\mathcal{C}^{w\heartsuit})_{C_2} \to \mathrm{GW}_0(\mathcal{C}^{w\heartsuit}, \Qoppa|_{\mathcal{C}^{w\heartsuit}}) \to L_0(\mathcal{C}, \Qoppa, w) \to 0$$

Moreover any strictly metabolic object $X \in \mathcal{C}^{w\heartsuit}$ is stably hyperbolic. That is for a Lagrangian $L \to X$ we have that there exists a Poincare object $Y \in \mathcal{C}^{w\heartsuit}$ such that $X \oplus Y$ is equivalent to $\mathrm{hyp}(L) \oplus Y$.

Proof. Follows from 6.12 by combining Theorem 8.17, Theorem 8.15 and Proposition 8.11. For the second part note that □

9. T-structures

Now we want to treat a second helpful device similar to weight structures, namely t-structures.

Definition 9.1. *A t-structure on a stable ∞-category $\mathcal{C}$ consists of two full, (saturated?) subcategories*

$$\mathcal{C}_{\tau\geq 0}, \mathcal{C}_{\tau\leq 0} \subseteq \mathcal{C}$$

such that the following axioms are satisfied:

1. *$\mathcal{C}_{\tau\geq 0}$ is closed under positive shifts and $\mathcal{C}_{\tau\leq 0}$ under negative shifts.*
2. *For $X \in \mathcal{C}_{\tau\geq 1} = \mathcal{C}_{\geq 0}[1]$ and $Y \in \mathcal{C}_{\tau\leq 0}$ we have that $[X, Y] = 0$.*
3. *For every X there is a cofibre sequence*
$$X' \to X \to X''$$
with $X' \in \mathcal{C}_{\tau\geq 1}$ and $X'' \in \mathcal{C}_{\tau\leq 0}$.

The definition is in some informal sense dual to that of a weight structure. But we will see now that this makes a huge difference as everything in a t-structure is much more canonical and rigid. We will use similar terminology as for weight structures:

$$\begin{aligned}
&\mathcal{C}_{\tau\geq n} := \mathcal{C}_{\tau\geq 0}[n] \\
&\mathcal{C}_{\tau\leq n} := \mathcal{C}_{\tau\leq 0}[n] \\
&\mathcal{C}_{\tau\in[a,b]} := \mathcal{C}_{a\leq\tau\leq b} := \mathcal{C}_{\tau\geq a} \cap \mathcal{C}_{\tau\leq b} \\
&\mathcal{C}^{\tau\heartsuit} := \mathcal{C}_{\tau\in[0,0]} \ .
\end{aligned}$$

The latter full subcategory will be refered to as the *heart of* $\mathcal{C}$. The t-structure will be called *bounded* if every object $X \in \mathcal{C}$ lies in $\mathcal{C}_{\tau\in[a,b]}$ for finite a and b.

Example 9.2. Let R be a ring. Then we define a t-structure on the derived ∞-category as follows: the connective objects $\mathcal{D}(R)_{\geq 0}$ are the connective chain complexes, that is those chain complexes $X \in \mathcal{D}(R)$ with $H_i(X) = 0$ for $i < 0$. The truncated ones $\mathcal{D}(R)_{\leq 0}$ are the chain complexes with $H_i(X) = 0$ for $i > 0$. For the mapping property we observe that if X is concentrated in degrees ≥ 1 then we have a projective replacement which is also concentrated in positive degrees. Thus the claim easily follows.

We then consider for every $X \in \mathcal{D}(R)$ represented as

$$... \to X_2 \to X_1 \xrightarrow{d_0} X_0 \to X_{-1} \to ...$$

the objects

$$X' = \Big(... \to X_2 \xrightarrow{d_1} \ker(d_0) \to 0 \to 0 \to ...\Big)$$

and

$$X'' = (... \to 0 \to 0 \to X_0/\mathrm{Im}(d_0) \to X_{-1} \to ...) \ .$$

Then there is a the desired fibre sequence $X' \to X \to X''$.

The heart of this t-structure is given by the ordinary category of R-modules Mod_R (no further condition like projective).

Example 9.3. In general there is no analogue of the t-structure of the previous example on the perfect derived category $\mathcal{D}^{\mathrm{perf}}(R) \subseteq \mathcal{D}(R)$ since the chain complexes X' and X'' from the factorization are in general not perfect. The problem is that the homology if a perfect chain complex is in general not perfect. But recall that a ring has global (projective) dimension $\leq d$ if every R-module M has a projective resolution of length at most d, that is

$$(\ldots \to 0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M)\,.$$

When R is in addition Noetherian and M is finitely generated then one can find such a resultion where the P_i are even finitely generated. In this case we have a t-structure with

$$\mathcal{D}^{\mathrm{perf}}(R)_{\geq 0}, \mathcal{D}^{\mathrm{perf}}(R)_{\leq 0} \subseteq \mathcal{D}^{\mathrm{perf}}(R)$$

being the respective connective and coconnective ones. Axioms (1) and (2) are clearly satisfied and for the third we have to argue that

$$X' = \left(\ldots \to X_2 \xrightarrow{d_1} \ker(d_0) \to 0 \to 0 \to \ldots\right)$$

is perfect. If we assume that X is strictly perfect, then all the X_i are finitely generated projective and by a filtration argument it suffices to see that $\ker(d_0)[1]$ is perfect. But then this follows since can be replace by a strictly perfect one.

The heart of this t-structure is given by the ordinary category of finitely generated R-modules (not necessarily projective).

Example 9.4. Let R be a connective ring spectrum. Then there is a t-structure on Mod_R given by $(\mathrm{Mod}_R)_{\geq 0}$. If $\pi_0(R)$ is Noetherian of finite global dimension and compact as an R-module then this induces a t-structure on Mod_R^{ω}. This is for example the case for $R = \mathrm{ku}$. The latter t-structure is however not bounded. For boundedness we need R to be truncated.

Definition 9.5. *We say that a weight structure on a stable ∞-category is (left) adjacent to a t-structure if $\mathcal{C}_{w\geq 0} = \mathcal{C}_{\tau\geq 0}$.*

Example 9.6. In the case $\mathcal{D}(R)$ and $\mathcal{D}^{\mathrm{perf}}(R)$ for R of Noetherian of finite global dimension the (standard) t-structure and weight structure are adjacent. There is always a functor $\mathcal{C}^{w\heartsuit} \to \mathcal{C}^{t\heartsuit}$ given by $P \mapsto \tau_{\leq 0}P$. In this case this is a full inclusion, but in general this need not happen (for example consider modules over a CDGA $R[x]/x^2$ with x in degree 1).

Lemma 9.7. *We have equalities*

$$\mathcal{C}_{\tau\geq 0} = \{B \in \mathcal{C} \mid [B, A] = 0 \textit{ for all } A \in \mathcal{C}_{\tau\leq -1}\}$$
$$\mathcal{C}_{\tau\leq 0} = \{A \in \mathcal{C} \mid [B, A] = 0 \textit{ for all } B \in \mathcal{C}_{\tau\geq 1}\}$$

Both of these subcategories are closed under all colimits and extensions.

Proof. Left to the reader as an exercise. □

Again this means that a t-structure is highly overdetermined like a weight structure. Now we come to the first real difference to a weight structure:

Lemma 9.8. *For $X \in \mathcal{C}_{\tau\geq 1} = \mathcal{C}_{\geq 0}[1]$ and $Y \in \mathcal{C}_{\tau\leq 0}$ we have that* $\mathrm{Map}_{\mathcal{C}}(X, Y) = 0$ *(the mapping space, not the mapping spectrum). Moreover the inclusion $\mathcal{C}_{\tau\geq n} \subseteq \mathcal{C}$ admit right adjoints*

$$\tau_{\geq n} : \mathcal{C} \to \mathcal{C}_{\tau\geq n}$$

and the inclusions $\mathcal{C}_{\tau\leq n} \subseteq \mathcal{C}$ admit left adjoints

$$\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\tau\leq n}$$

so that the sequence

$$\tau_{\geq n+1}X \to X \to \tau_{\leq n}X$$

with its unique nullhomotopy (by the first assertion) becomes a fibre sequence. In fact every other fibre sequence $X' \to X \to X''$ with $X' \in \mathcal{C}_{\tau \geq n+1}$ and $X'' \in \mathcal{C}_{\tau \leq n}$ is canonically equivalent to this one.

Proof. The first statement about the mapping space follows immediately from axiom (2) by shifting source or target. For the right adjoint we simply choose $X' \to X$ such that X' is in $\tau \geq n$ and the cofibre is in $\tau_{<n}$. Then one directly gets that for every object $Y \in \mathcal{C}_{\geq n}$ the space of maps from Y to X is equivalent to the space of maps from Y to X' as the the space of maps from Y to X/X' is contractible. The rest follows similary and is left as an exercise. □

For a given object $X \in \mathcal{C}$ we shall write

$$\pi_n^{\heartsuit}(X) := \tau_{\geq 0}\tau_{\leq 0}X[-n] \in \mathcal{C}^{\heartsuit} \ .$$

One should generally see t-structures as a way of making homotopy groups and Postnikov towers precise. In contrast to weight structures which made cell structures precise. The analog of the cell decomposition of an object os the Postnikov tower

$$... \to \tau_{\geq n+1}X \to \tau_{\geq n}X \to \tau_{\geq n-1}X \to ... \qquad \to X$$

for every object $X \in \mathcal{C}$ which is finite for every X iff the weight structure is bounded. The subquotients lie in shifts of the heart (and are given by the homotopy groups). The advantage here is that the Postnikov tower is fully functorial.

Lemma 9.9. *For a t-structure the heart $\mathcal{C}^{\tau\heartsuit}$ is an ordinary abelian category.*

Proof. By the truncatedness of the mapping spaces the heart is an ordinary category. The heart is closed under finite biproducts, thus it is additive. It has kernels and cokernels which are given as follows:

$$\operatorname{coker}(A \to B) = \tau_{\leq 0}(\operatorname{cof}(A \to B)) \ .$$

and dually for kernels. Then a monomorphism is precisely a morphism such that the cofibre is 0-truncated and an epimorphism is a morphism such that the fibre is connective. It follows that a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

with $A \to B$ mono and $B \to C$ epi is a pullback in the heart if it is a pushout. □

Remark 9.10. If $\mathcal{C}$ admits a bounded t-structure then it is idempotent complete. To see this assume that we have an idempotent $e : X \to X$. We get induced idempotents $\tau_{\geq 0}e$ and $\tau_{\leq 0}e$ and e splits precisely if $\tau_{\geq 0}(e)$ and $\tau_{\leq 0}(e)$ do. Then using this one can inductively shows that e splits precisely of $\pi_n(e)$ splits for each n. But this is a morphism in the heart and thus in an abelian category which is clearly idempotent complete (as here splitting an idempotent is a finite limit in contrast to the ∞-categorical situation).

Definition 9.11. *For an abelian category $\mathcal{A}$ we define the K-theory as*

$$K_0(\mathcal{A}) = \frac{\{\textit{Iso classes of objects in } \mathcal{A}\}}{\{[B] = [A] + [C] \textit{ for a short exact sequence } A \to B \to C.\}}$$

Proposition 9.12. *For a bounded t-structure the map*

$$K_0(\mathcal{C}^{\tau\heartsuit}) \to K_0(\mathcal{C}) \qquad [A] \mapsto [A[0]]$$

is an isomorphism.

Proof. The morphism is clearly well-defined. We have an inverse morphism given by $X \mapsto \sum(-1)^n \pi_n^{\heartsuit}(X)$. This is also well-defined (by the long exact seuqence). One composite is clearly the identity and the other one by the existence of finite Postnikov towers. $\square$

Remark 9.13. Assume that $\mathcal{C}$ admits adjacent bounded weight and t-structures. Then we get an isomorphism

$$K_0(\mathcal{C}^{w\heartsuit}) \to K_0(\mathcal{C}^{t\heartsuit}) \quad P \mapsto \sum(-1)^n \pi_n^{\heartsuit}(P) \ .$$

In the case of an ordnary ring it sends $[P]$ to $[P]$. The inverse sends $M \in \mathcal{C}^{t\heartsuit}$ to $\sum(-1)^n P_n$ for a cell structure with subquotients P_i in the heart (i.e. a projective resolution in the ordinary case).

Example 9.14. If $\mathcal{C}$ has adjacent weight and t-structures then neither boundedness implies the other. For the 'standard' structures on $\mathrm{Sp}^{\mathrm{fin}}$ only the weight structure is bounded. Conversely consider the bounded derived category $\mathcal{D}^b(\mathbb{Z}[C_2])$. This has a bounded t-structure with heart $\mathrm{Mod}_{\mathbb{Z}[C_2]}$ and a weight structure with heart $\mathrm{Proj}_{\mathbb{Z}[C_2]}$. But the object $\mathbb{Z}$ does not admit a finite weight resolution.

Definition 9.15. *For a given t-structure on a stable ∞-category $\mathcal{C}$ we define the* global dimension *as*

$$\dim(\mathcal{C}, \tau) := \min\left\{d \mid \mathrm{map}_{\mathcal{C}}(X,Y) \textit{ is } (-d)\textit{-connective for } X,Y \in \mathcal{C}^{\tau\heartsuit}\right\}$$

if this number is finite or ∞ else.

Example 9.16. If a ring R (with involution) has global projective dimension d then

$$\dim(\mathcal{D}(R), \tau) = d$$

and the same for $\mathcal{D}^{\mathrm{perf}}(R)$ if R is noetherian. This follows since the projective dimension can be defined in terms of vanishing of Ext groups. For a field k we get that $\dim(\mathcal{D}(k), \Omega^s, \tau) = 0$. For $R = \mathbb{Z}$ we have $\dim(\mathcal{D}(\mathbb{Z}), \Omega^s, \tau) = 1$ as it has projective dimension 1. The dimension of ku with its standard t-structure is 4 (exercise).

Lemma 9.17. *Assume that $\mathcal{C}$ admits a weight structure compatible with the duality and adjacent to the t-structure. Then we have:*

$$\begin{aligned}\dim(\mathcal{C}, \tau) &= \min\left\{d \mid \mathcal{C}^{\tau\heartsuit} \subseteq \mathcal{C}_{w\in[0,d]}\right\} \\ &= \min\left\{d \mid D(\mathcal{C}^{\tau\heartsuit}) \subseteq \mathcal{C}_{\tau\geq -d}\right\}\end{aligned}$$

Proof. The second equality is clear by Lemma 8.13. For the first we note that if $X \in \mathcal{C}_{w\in[0,d]}$ and $Y \in \mathcal{C}_{w\geq 0}$ then

$$\mathrm{map}_{\mathcal{C}}(X,Y) = \mathrm{map}_{\mathcal{C}}(X[-d],Y)[d]$$

is $(-d)$-connective since $\mathrm{map}_{\mathcal{C}}(Y[-d],X)$ is connective. If conversely for a given X the spectrum $\mathrm{map}_{\mathcal{C}}(X,Y)$ is $-d$-connective the it follows that $[X[-d-1],Y] = 0$ for every $Y \in \mathcal{C}_{w\geq 0}$. Thus X is in $w \leq d$. $\square$

Now we want to perform a version of surgery in the setting of a (not-necessarily bounded) t-structure on a Poincaré-∞-category. We will do this first in the setting of a quadratic functor, i.e. $\Qoppa = \Qoppa^s_B$ for a symmetric bilinear functor.

Lemma 9.18. *Assume that a t-structure on $(\mathcal{C}, \Qoppa^s_B)$ is of global dimension d.*

(1) *Every Poincaré object (of dimension 0) is bordant to one with $\tau \geq -d/2$.*
(2) *Every Poincaré object of dimension $n \in \mathbb{Z}$ is bordant to one with $\tau \geq -\frac{d+n}{2}$.*

Proof. We first prove the first part: For a Poincaré object X consider the morphism

$$X \to \tau_{<-k} X$$

with $k = \frac{d}{2}$. We set $S := D(\tau_{\leq -k} X)$. By assumption this has connectivity

$$\tau > k - d = \tfrac{d}{2} - d = -k.$$

As a result we find that $\mathrm{Map}_{\mathcal{C}}(S, \tau_{<-k} X) = 0$. Thus

$$\Qoppa(S) = \mathrm{map}_{\mathcal{C}}(S, DS)^{hC_2} = \mathrm{map}_{\mathcal{C}}(S, \tau_{<-k} X)^{hC_2}$$

has vanishing π_0. As a result we can perform surgery on the morphism

$$s : S \to X$$

dual to $X \to \tau_{<-k} X$. But this surgery result in an object X_s given as the cofibre of

$$S \to \tau_{\geq -k} X$$

which is $(-k)$-connective.

For the second part we consider the Poincaré-∞-category $(\mathcal{C}, \Qoppa[-n])$ with the duality given by

$$X \mapsto (DX)[-n] \ .$$

This is also symmetric (with the shifted B) and of dimension $d+n$ so that the second part immediately follows from the first. $\square$

In particular this implies that if there is an adjacent weight structure compatible with the duality (i.e. the duality preserves the weight heart) then the resulting Poincaré object has weight in

$$w \in [-\tfrac{d+n}{2}, \tfrac{d-n}{2}] \ .$$

Equivalently this means that is has weight length $\leq d+1$ and is centered around the middle dimension, which might be easier to remember. We make this precise in the following definition:

Definition 9.19. *A Poincaré object X of dimension in $(\mathcal{C}, \Qoppa)$ is said to have lenght $\leq k$ if it lies in the weight intervall*

$$w \in [\tfrac{-n-k+1}{2}, \tfrac{-n+k-1}{2}]$$

Similarly a Lagrangian is said to have length $\leq k$ if it lies in the same weight interval.

Note that depending on the parity of $n+k$ this might actually mean length k or $k-1$ centered around the middle dimension.

Corollary 9.20. *Assume $\mathcal{C}$ admits adjacent weight and t-structures, the dimension of $(\mathcal{C}, \tau)$ is zero and the duality preserves the weight heart. Then we have that $L_{odd}(\mathcal{C}, \Qoppa^s_B) = 0$. In particular the odd symmetric L-theory $L^s_*(k)$ of fields vanishes.*

The even dimensional groups are given by the respective Witt groups and the isomorphism is implemented by the map

$$L_{2n}(\mathcal{C}.\Omega^s_B) \mapsto W_{2n}(\mathcal{C}, \Omega^s_B) \qquad C \mapsto H_n(C)$$

Proof. For every n we get that the objects can be represented by objects with

$$w \in [-\tfrac{n}{2}, -\tfrac{n}{2}] \ .$$

For n odd this intervall is empty. □

The even L-groups in the last example can be represented by objects in a single weight. The question is whether they agree with the respective Witt groups. The question is a question about Lagrangians and we have the following result.

Lemma 9.21. *Assume that $\mathcal{C}$ admits adjacent weight and t-structures and that the dimension of $(\mathcal{C}, \tau)$ is d. If a Poincaré object X of weight length $\leq d+1$ admits a Lagrangian, then there exist Poincaré objects M, M' of weight length $\leq d+1$ which admit Langrangians of weight length $\leq d+1$ such that*

$$X \oplus M \simeq M' \ .$$

Proof. First we perform surgery from above as in Lemma 9.18 to reduce a given Lagrangian L to have $\tau \geq -\frac{d+n}{2} - 1$. Then we need to kill the bottom weight cell as in Section 7. Note that for this last step no assumption on Ω was necessary. □

Proposition 9.22. *Assume $\mathcal{C}$ admits adjacent weight and t-structures, the dimension of $(\mathcal{C}, \tau)$ is d and the duality preserves the weight heart. Then we have that*

$$L_n(\mathcal{C}, \Omega^s_B) = \frac{\{\textit{Poincaré objects of weight length } d+1\}}{\{\textit{Those with Lagrangians of weight length } d+1\}} \ .$$

Corollary 9.23. *Assume $\mathcal{C}$ admits adjacent weight and t-structures, the dimension of $(\mathcal{C}, \tau)$ is ≤ 1 and the duality preserves the weight heart. Then for every n the morphism*

$$W_n(\mathcal{C}, \Omega, w) \to \mathrm{L}_n(\mathcal{C}, \Omega^s_B)$$

is an isomorphism.

Corollary 9.24. *For $R = \mathbb{Z}$ or more generally a Dedekind ring we have that*

$$L^s_n(R) = W^s_n(R) \ .$$

Also recall that in the quadratic case we found that $L^q_n(R) = W^q_n(R)$ for all n.

The upshot of the previous discussion is that the symmetric L-groups can somehow be understood, especially if the dimension d is small. The question is now what we can do about an arbitrary Ω in the presence of adjacent weight and t-structures. We have already seen that if Ω is connective on the weight heart then we can understand the L-groups until dimension ≤ 1 as the Witt groups. The higher ones where described in terms of longer complexes and thus remain somewhat mysterious (especially since the lenght tends to ∞ with n and does not remain fixed).

We would like to use t-structures to simplify the situation here as well. To this end we make the following definition

Definition 9.25. *Let $(\mathcal{C}, \Omega)$ be a Poincaré-∞-category with a t-structure. We say that the two structures are compatible if the fibre*

$$L^{\Omega}(X) = \mathrm{fib}(\Omega(X) \to B_{\Omega}(X,X)^{hC_2}$$

is a coconnetive spectrum (i.e. in $\mathrm{Sp}_{\geq 0}$) for each $X \in \mathcal{C}_{\geq 0}$.

Remark 9.26. Assume that the duality of $\mathcal{C}$ sends $\mathcal{C}_{\geq 0}$ to $\mathcal{C}_{\leq 0}$. Then a t-structure is compatible precisely if Ω sends connective objects to coconnective spectra. This follows easily from the cofibre sequence

$$L^{\Omega}(X) \to \Omega(X) \to \mathrm{map}_{\mathcal{C}}(X, DX)^{hC_2}$$

using that $\mathrm{map}_{\mathcal{C}}(X, DX)$ is connective in this case. If there is an adjacent weight structure then the condition that $D(\mathcal{C}_{\geq 0}) \subseteq \mathcal{C}_{\leq 0}$ is equivalent to the assertion that $\mathcal{C}_{w \leq 0} \subseteq \mathcal{C}_{\tau \leq 0}$. In particular the weight heart is contained in the t-heart and thus is an ordinary category. This is of course satisfied for for derived categories of rings but not for ring spectra.

If the t-structure is moreover bounded then it is compatible with Ω precisely if Ω sends the heart to coconnective spectra.

Example 9.27. The symmetric functor Ω^s_B is compatible with the t-structure if D sends $\mathcal{C}_{\geq 0}$ to $\mathcal{C}_{\leq 0}$. The quadratic functor is generally not compatible with the usual t-structures (unless we are in a situation where it agrees with the symmetric one).

Theorem 9.28. *Assume $\mathcal{C}$ admits adjacent weight and t-structures*[16] *and a Poincaré-structure Ω such that the duality preserves the weight heart and such that Ω is compatible with the t-structure. We denote the dimension of $(\mathcal{C}, \Omega, \tau)$ by d. Then we have that*

$$L_n(\mathcal{C}, \Omega^s_B) = \frac{\{\textit{Poincaré objects of lenght } d+1\}}{\{\textit{Those with Lagrangians of length } d+1\}} \ .$$

for $n \geq \dim(\mathcal{C}, \Omega, \tau)$.

Proof. We will verbatim carry the proofs of Lemma 9.18 and Lemma 9.21 over. One simply needs to work out that the surgery still works: for an n-dimensional Poincaré object we want to perform surgery on the morphism

$$s : S \to X$$

with $S = D(\tau_{<-k}X)[-n]$ and $k = \frac{d+n}{2}$. Thus we need to verify that $\pi_n(\Omega(S)) = 0$. This homotopy group sits in an exact sequence

$$\pi_n(L^{\Omega}(S)) \to \pi_n(\Omega(S)) \to \pi_n(\mathrm{map}_{\mathcal{C}}(S, DS)^{hC_2}) \ .$$

The latter term is zero since

$$\pi_n(\mathrm{map}_{\mathcal{C}}(S, DS)^{hC_2}) = \pi_0(\mathrm{map}_{\mathcal{C}}(S, \tau_{<-k}X)^{hC_2})$$

and S has connectivity $\tau > (k - d - n) = -\frac{d+n}{2} = -k$. Thus it suffices to verify that the first term $\pi_n(L^{\Omega}(S))$ is zero. By assumption of compatiblity as in Definition 9.25 we find that $L^{\Omega}(S)$ has $\tau < k$. Now the claim follows as soon as $n \geq k$. This is equivalent to

$$n \geq d = \dim(\mathcal{C}, \Omega, \tau) \ .$$

[16]The weight structure is not really necessary. In presence of a t-structure one can also express a similar result in terms of connectivities. But we find the formulation in terms of weight intervals more useful and also the last surgery step for Lagrangians requires it.

For Lagrangians the argument works the same. □

Corollary 9.29. *If $\mathcal{C}$ admits adjacent weight and t-structures and the t-structure is compatible. Then for any quadratic functor Ω the L-groups $L_n(\mathcal{C}, \Omega)$ agree with the symmetric ones (i.e. with $L_n(\mathcal{C}, \Omega^s_{B_\Omega})$) for $n \geq \dim +3$. In particular in the case of Noethering rings of finite global dimension they become 4-periodic above the critical dimension.*

Proof. We have a canonical map $L_n(\mathcal{C}, \Omega) \to L_n(\mathcal{C}, \Omega^s)$. Using Theorem 9.28 we need to argue that the functors $\tau_{\geq n}\Omega$ and $\tau_{\geq n}\Omega^s$ agree on objects X with $\tau \geq -\frac{n+d}{2}$ as long as $n > d+2$. Using the fibre sequence

$$L^{\Omega}(X) \to \Omega(X) \to \Omega^s(X)$$

it suffices to show that $\pi_*(L^{\Omega}(X))$ vanishes for $* \geq n-1$. But we have that

$$L^{\Omega}(X) = L^{\Omega}\left(X[\tfrac{n+d}{2}]\right)[\tfrac{n+d}{2}]$$

is $(\frac{n+d}{2})$-truncated. Thus we need that

$$n - 1 > \tfrac{n+d}{2}$$

which is equivalent to $n > d+2$. This shows the bijectivity. □

Remark 9.30. A variant of this argument shows that in degree $n = \dim +2$ the canonical map $L_n(\mathcal{C}, \Omega) \to L_n(\mathcal{C}, \Omega^s)$ is injective.

Now assume that we have adjacent weight and t-structures compatible with the duality. Our results about the quadratic and symmetric functors show that for the quadratic functor one gets that $L_n(\mathcal{C}, \Omega)$ can in all degrees be descibed in terms of objects of lenfth ≤ 2. For the symmetric functor it can be described in alle degrees as objects of weight length $d+1$. Now we want to place ourselves in a setting in between those functors.

Definition 9.31. *Assume that $\mathcal{C}$ admits adjacent weight and t-structures of dimension d and a Poincaré structure (not necessarily compatible). Then we define*

$$L_n(\mathcal{C}, \Omega, w, \tau) = \frac{\{\textit{Poincaré objects of lenght } k(n)\}}{\{\textit{Those with Lagrangians of length } k(n)\}}$$

where

$$k(n) = \begin{cases} 2 & \text{for } n \leq 1 \\ n+1 & \text{for } 1 \leq n \leq d \\ d+1 & \text{for } n \geq d \end{cases}$$

Theorem 9.32. *Assume that $(\mathcal{C}, \Omega)$ admits adjacent weight and t-structures and a Poincaré structure compatible with both. Then we have that*

$$L_n(\mathcal{C}, \Omega, w, \tau) \to L_n(\mathcal{C}, \Omega)$$

is an isomorphism.

Note that in the case $d = 0$ this does not make a difference in degree 0 as length 2 in fact also means lenght 1 due to the fact that the intervall would be $[-\frac{1}{2}, \frac{1}{2}]$.

Lemma 9.33. *Assume that a Poincaré ∞-category $(\mathcal{C}, \Omega)$ admits adjacent bounded weight and t-structures such that $\mathcal{C}^{w\heartsuit} \subseteq \mathcal{C}^{\tau\heartsuit}$. Then Ω is compatible with both precisely if it sends the weight heart $\mathcal{C}^{w\heartsuit}$ to the heart $\mathrm{Sp}^{\tau\heartsuit} \simeq \mathrm{Ab}$.*

Proof. According to the definitions and Remark 9.26 we have to verify that if $Ϙ$ sends the weight heart to connective spectra then the following are equivalent:

(1) $Ϙ$ sends the t-heart to coconnective spectra
(2) $Ϙ$ sends the weight heart to coconnective spectra.

Using the fibre sequence

$$L^{Ϙ}(X) \to Ϙ(X) \to \mathrm{map}_{\mathcal{C}}(X, DX)^{hC_2}$$

we can verify that the two similar assertions with $Ϙ$ replaced by $L^{Ϙ}$ are equivalent. But then by resolving objects in the heart this easily follows. □

Corollary 9.34. *Assume a Poincaré-∞-category $(\mathcal{C}, Ϙ)$ admits adjacent weight and t-structures, the dimension is 1 and the duality is compatible with the weight structure. Then the morphism*

$$W_n(\mathcal{C}, Ϙ, w) \to L_n(\mathcal{C}, Ϙ)$$

is an isomorphism for all n under one of the following assumptions:

(1) *The quadratic functor $Ϙ$ is homogenous, i.e. $Ϙ = Ϙ^q_B$.*
(2) *The quadratic functor $Ϙ$ is cohomogenous, i.e. $Ϙ = Ϙ^s_B$.*
(3) *The quadratic functor is compatible with the weight and the t-strucutre.*

Proof. Clear using the definition of the Witt group. □

Remark 9.35. The Witt group $W_0(\mathcal{C}, Ϙ, w)$ only depends on $\mathcal{C}^{w^\heartsuit}$ and $\pi_0(Ϙ|_{\mathcal{C}^{w^\heartsuit}})$. It has generators given by objects $X \in \mathcal{C}^{w^\heartsuit}$ together with an element $q \in \pi_0(Ϙ(X))$ such that the adjunct map $X \to DX$ is an equivalence.

10. Computations of L-groups

Now we want to use our previous discussions to compute L-groups. We start with the case of fields k. If the characteristic of k is different from 2 then the quadratic and symmetric L-groups agree and we shall simply write $L_*(k)$ for those. The same applies for the Witt group which we simply write as $W(k)$.

Proposition 10.1. *Let k be a field of characteristic different from 2. Then we find that $L_*(k) = W_0(k)$ for $* = 4n$ and $L_*(k) = 0$ else. The isomorphism is implemented by*

$$L_{4n}(k) \to W_0(k) \qquad C \mapsto H_{2n}(C) \ .$$

where $H_{2n}(C)$ is equipped with the form of Construction 5.3 .

Proof. We already now that the groups are 4-periodic, that they vanish in odd degrees and agree with the respective Witt groups from Corollary 9.20 and Corollary 9.34. Thus we need to show that the Witt group $W_2(k)$ vanishes. This is the Witt group of antisymmetric bilinear forms:

Let V be a vector space with such an antisymmetric form β. If V admits a non-trivial vector v then we find that $\beta(v, v) = 0$ by antisymmetry and $\frac{1}{2} \in k$. Since β is non-degenerate we find another vector $w \in V$ such that $\beta(v, w) = 1$. We set $V_0 = \langle v, w\rangle \subseteq V$ and get an orthogonal decomposition

$$V = V_0 \oplus V_0^{\perp} \ .$$

To see this note that for an element $x \in V$ the element

$$x - \beta(x, w)v - \beta(w, w)\beta(x, v)v + \beta(x, v)w$$

lies in $V_0^\perp$. But $\langle v\rangle$ is a Lagrangian in V_0 so that in the Witt group we have that $[V] = [V_0^\perp]$. Inductively this show the claim.

Finally the morphism

$$L_{4n}(k) \to W_0(k) \qquad C \mapsto H_{2n}(C)\ .$$

is well-defined since it preserves Poincaré objects and for a Lagrangian in C we get an induced Lagrangian. The composition with the isomorphism $W_0(k) \to L_{4n}(k)$ is clearly the identity. $\square$

For any element $u \in k^\times$ we get a form on the 1-dimensional k-vector space represented by u. We denote the image of this element in $W(k)$ be $[u]$.

Proposition 10.2. *Let k of characteristic $\neq 2$. Then $W(k)$ is generated by elements $[u]$ for $u \in k^\times$. If k is additionally quadratically closed we have that the morphism*

$$\dim : W(k) \to \mathbb{Z}/2$$

is an isomorphism. For $k = \mathbb{R}$ we have that

$$\mathrm{sgn} : W_0(\mathbb{R}) \to \mathbb{Z}$$

is an isomorphism.

Proof. Given any vector space V with a symmetric, non-degenerate form. There is a vector with $\beta(v, v) \neq 0$ since otherwise β would vanish by the polarization formula. We set $V_0 = \langle v\rangle$ and get that $V = V_0 \oplus V_0^\perp$ and continue inductively. This gives an orthogonal basis and shows the generators description.

In the case of a quadratically closed field we claim that the form on $k \oplus k$ given by the diagonal matrix with entires u and u' for $u, u' \in k^\times$ is metabolic. To this end we observe that the vector

$$\left(1, \sqrt{-\frac{u}{u'}}\right)$$

is isotropic and thus defines a Lagrangian.

Finally by Sylvester's theorem the signature is well defined on the Witt group. It is an isomorphism since $[u] = [\pm 1]$. $\square$

Corollary 10.3. *If k is a quadratically closed field of characteristic $\neq 2$ then*

$$L_{4n}(k) \to \mathbb{Z}/2 \qquad [C] \mapsto [\dim H_{2n}(C)]$$

is an isomorphism. The map

$$L_{4k}(\mathbb{R}) \mapsto \mathbb{Z} \qquad [C] \mapsto [\mathrm{sgn}\, H_{2n}(C)]$$

is an isomorphism.

Proposition 10.4. *For the complex numbers $\mathbb{C}$ with its canonical involution we find that*

$$L_{2n}(\mathbb{C}, \overline{-}) \to \mathbb{Z} \qquad C \mapsto \mathrm{sgn} H_n(\mathbb{C})$$

is an isomorphism and $L_{odd}(\mathbb{C}, \overline{-}) = 0$.

Proof. The odd groups vanish by our general results for 0-dimensional Poincaré categories. We claim that the groups are 2-periodic. This follows since hermitian forms β with $\beta(v, w) = \overline{\beta(w, v)}$ and antihermitian forms with $\beta(v, w) = -\overline{\beta(w, v)}$ are in 1-1 correspondence by $\beta \mapsto i\beta$. Another way of saying that is that $(\mathcal{D}^{\mathrm{perf}}(\mathbb{C}), \Qoppa)$ and $(\mathcal{D}^{\mathrm{perf}}(\mathbb{C}), \Qoppa[2])$ are equivalent. Thus we are left to compute L_0. But this again follows by Sylvester's theorem. $\square$

Remark 10.5. One can more generally show that for any complex, unital C^*-algebra A one has a natural isomorphism $L_*(A,*) \cong K_*^{\mathrm{top}}(A)$.

In the case that $\mathrm{char}(k) \neq 2$ we actually have that $W(k)$ is a ring and that it has a canonical ideal $I \subseteq W(k)$ given by the kernel of $\dim : W(k) \to \mathbb{Z}/2$.

Theorem 10.6 (Voevodsky, Milnor Conjecture)**.** *The associated graded of this filtration is given by*

$$I^{n+1}/I^n = K_M^n(k)/2 = H^n_{\acute{e}t}(k, \mathbb{F}_2)\ .$$

Now we want to investigate the case of fields of characteristic 2. In this case we have to ditinguish between the symmetric and the quadratic L-groups $L_*^s(k)$ and $L_*^q(k)$.

Proposition 10.7. *Let k be a field of characteristic 2. Then we have*

$$L_*^s(k) = \begin{cases} W_0^s(k) & \text{for } * \text{ even} \\ 0 & \text{for } * \text{ odd} \end{cases} \qquad L_*^q(k) = \begin{cases} W_0^q(k) & \text{for } * \text{ even} \\ 0 & \text{for } * \text{ odd}\ . \end{cases}$$

Proof. The 2-periodicity is shown in Proposition 6.5 and the symmetric case follows from Theorem 9.20. The fact that $L_0^q(k) = W_0^q(k)$ follows from Proposition 7.11. Thus it only remains to show that $L_1^q(k)$ vanishes. We already know that any Poincaré object can be assumed to lie in degrees $[-1,0]$ (here weight and τ agree). In this case however we can still perform surgery: consider a split map $H_{-1}(X)[-1] \to X$. Then we have that $\pi_1(H_{-1}(X)[-1]) = 0$ so that we can perform surgery. The resulting object is given as the homology of

$$H_{-1}(X)[-1] \to X \to D(H_{-1}(X))[0] = H_0(X)[0]\ .$$

This homology is zero since this is a cofibre sequence. □

Proposition 10.8. *The maps*

$$\mathrm{Arf} : W_0^q(\mathbb{F}_2) \to \mathbb{Z}/2 \quad \text{and} \quad \dim : W_0^s(\mathbb{F}_2) \to \mathbb{Z}/2$$

are isomorphisms and the canonical map

$$W_0^q(\mathbb{F}_2) \to W_0^s(\mathbb{F}_2)$$

is zero.

We recall that the Arf invariant (in its easiest form) is an invariant for non-degenerate quadratic spaces (V,q) over $\mathbb{F}_2$. It s also called the democratic invariant. and is given by the value of $\mathbb{F}_2$ that occurs more often among the values of the quadratic form $q(v)$ as v ranges through V, another (equivalent) way of writing that is as

$$\mathrm{Arf}(V,q) = \frac{1}{2^{\dim V/2}}\Big(\#\{v \in V \mid q(v) = 0\} - \#\{v \in V \mid q(v) = 1\}\Big) \in \{\pm 1\} \tag{5}$$

There is an implicit claim that this procedure always comes to a conclusion which is a priori not clear and that the latter desceription always lies in $\{\pm 1\}$. We will verify both claims in the proof.

Proof. For the symmetric case we first note that as in Proposition 10.1 for every vector $v \in V$ with $\beta(v,v)$ we can split off a 2-dimensional metabolic space. Thus we can assume that for a given symmetric space (V,β) over $\mathbb{F}_2$ we have that $\beta(v,v) = 1$

for each $v \in V$. But then we are at most 1-dimensional since for every vector $w \neq v, 0$ we would have that

$$1 = \beta(v+w, v+w) = \beta(v,v) + 2\beta(v,w) + \beta(w,w) = 0 \ .$$

The 1-dimensional space $\mathbb{F}_2$ with the form $(x,y) \mapsto xy$ exists and is detected by the invariant dimension modulo 2 (which is obviously well-defined).

Now assume that (V,q) is a quadratic space over $\mathbb{F}_2$. For the associated symmetric bilinear form $\beta(v,w) = q(v+w) - q(v) - q(w)$ we have that $\beta(v,v) = q(2v) - 2q(v) = 0$ so that the underlying symmetric space is even dimensional and the class in the symmetric Witt group vanishes.

If $q(v) = 0$ for some v then we can split off a 2-dimensional metabolic space as follows: as before we pick w with $\beta(v,w) = 1$ and then form the orthogonal complement. This is metabolic since v is a Lagrangian. In fact it is hyperbolic: a priori it is not true that $q(w) = 0$. But if $q(w) = 1$ then we replace w by $w+v$ and get that

$$q(w+v) = q(w) + q(v) + \beta(v,w) = 1 + 0 + 1 = 0'.$$

Using this procedure to split off hyperbolic spaces we can therefore assume that $q(v) = 1$ for each $v \in V$. Assume that $v \neq w$ are non-trivial elements of V, then the associated bilinear form is given by

$$\beta(v,w) = q(v+w) - q(v) - q(w) = 1$$

Thus for a third $z \in V \setminus \{0, v, w\}$ we get that

$$\beta(v+z, w) = \beta(v,w) + \beta(z,w) = 1 + 1$$

so that $v + z = w$. Thus the dimension of V can at most be 2. And indeed the form

$$q : V = (\mathbb{F}_2)^2 \to \mathbb{F}_2$$

that sends each non-trivial element to 1 is indeed quadratic since it can be written as

$$q(v) = v_1^2 + v_1 v_2 + v_2^2 \ .$$

As a result the Witt group is generated by this single element V and we want to determine its order. Clearly $V \oplus V$ admits the diagonal V as a Lagrangian (since it is isotropic and of half the dimension). Thus $2[V] = 0$. In order to show that $[V]$ is non-trivial it suffices to show that the Arf invariant defined as in (5) indeed is a well-defined homomorphism

$$\mathrm{Arf} : W_0^q(\mathbb{F}_2) \to (\mathbb{Q}, \cdot)$$

since $\mathrm{Arf}(V) = 1$. It is clear that Arf is additive by a direct verification. Thus we have to see that it indeed vanishes on metabolics. But this is also clear since every metabolic is a sum of hyperbolics by the above argument splitting off hyperbolics and for a 2-dimensional hyperbolic H we have that $\mathrm{Arf}(H) = \frac{1}{2}(3-1) = +1$.

Finally the claims about the Arf invariant also follow. $\square$

Remark 10.9. In fact a slight elaboration of the above arguments proves that any non-degenerate symmetric form over $\mathbb{F}_2$ is isomorphic to one of

$$H^n, H^n \oplus \mathbb{F}_2, H^n \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$$

where $H = \mathrm{hyp}(\mathbb{F}_2)$ and $\mathbb{F}_2$ carries the form $(x,y) \mapsto xy$. Every non-degenerate quadratic form over $\mathbb{F}_2$ is isomorphic to

$$H^n, H^n \oplus V$$

where $V = (\mathbb{F}_2)^2$ is given by $q(v) = 1$ for all $v \neq 0$ and H is the hyperbolic from on $(\mathbb{F}_2)^2$ given by $q(v) = v_1 v_2$.

Corollary 10.10. *We have the isomorphisms in L-theory given by*

$$L^q_{2n}(\mathbb{F}_2) \to \mathbb{Z}/2 \qquad [C] \mapsto [\mathrm{Arf} H_n(C)]$$

and

$$L^s_{2n}(\mathbb{F}_2) \mapsto \mathbb{Z}/2 \qquad [C] \mapsto [\dim H_n(C)]$$

is an isomorphism. given by

Proposition 10.11. *Assume that k is a general field of characteristic 2, then $W^s_0(k)$ is generated by 1-dimensional spaces. If k is quadratically closed then $W^s_0(k) = \mathbb{Z}/2$ given by the dimension. If k is even algebraically closed then $W^q_0(k) = 0$.*

Proof. Let V be a symmetric space over k. As in the proof of Proposition 10.8 we can assume that $\beta(v, v) \neq 0$ for each $v \in V \setminus 0$. Then we can inductively find an orthogonal basis so that we see that the Witt ring is generated by one dimensional symmetric spaces. Clearly two 1-dimensional spaces (k, α) and (k, β) are equivalent if α and β differ by a square, which is always the case in a quadratically closed field. Moreover for every space V the space $V \oplus V$ has V as a Lagrangian, so that it is 2-torsion.

Over algebraically closed fields any two quadratic forms of the same dimension are isomorphic. This implies the last claim since the dimension has to be even. □

Now we turn to the case of the integers. We have the following result:

Theorem 10.12. *The L-groups of the integers are given by*

$$L^q_*(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 4n \\ \mathbb{Z}/2 & \text{for } * = 4n+2 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad L^s_*(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 4n \\ \mathbb{Z}/2 & \text{for } * = 4n+1 \\ 0 & \text{else} \end{cases}$$

The canonical map $L^q_*(\mathbb{Z}) \to L^s_*(\mathbb{Z})$ is given by multiplication with 8 in degree $4n$ and zero else. Signature mod 8, Kervaire, deRham invariant, Signature

We will prove this result now. We start with degree 0. First observe that a quadratic form over $\mathbb{Z}$ is uniquely detemined by its associated symmetric bilinear form β via the fomula

$$q(m) = \tfrac{\beta(m,m)}{2} \ .$$

Thus it is a property of symmetric bilinear forms to be quadratic. This is the case precisely if $\beta(m, m)$ is even for each $m \in M$. In this case the form is also called even and otherwise odd. This behaviour is also called the *type* of a symmetric bilinear form. Moreover a form is called definite if it is either positive or negative definite, otherwise indefinite. We will use the following result (see for example Serre's book Cours d'arithmétique for a proof).

Theorem 10.13 (Hasse-Minkowski)**.** *Two unimodular, indefinite forms over $\mathbb{Z}$ are isomorphic precisely if they have the same dimension, signature and type.*

Proposition 10.14. *The signature morphism*

$$L^s_{4n}(\mathbb{Z}) \to L^s_{4n}(\mathbb{R}) \to \mathbb{Z} \qquad C \mapsto \mathrm{sgn}\,(H_{2n}(C) \otimes \mathbb{R})$$

is an isomorphism.

Proof. We first compute $W_0^s(\mathbb{Z})$. Given any two unimodular symmetric bilinear forms over $\mathbb{Z}$ of the same signature. We want to show that they represent the same element in the Witt group.

We can add a hyperbolic forms to make both indefinite without changing the classes in the Witt group (the hyperbolic form is clearly indefinite). We observe that $\dim - \mathrm{sgn}$ is always twice the number of negative diagonal entries, in particular always even. As a result we find that our two forms can be made of equal dimension by adding hyperbolic forms. Now both of our forms become isomorphic after adding the one dimensional trivial form, which makes them odd.

This shows injectivity of the signature morphism. Surjectivity is clear. □

Proposition 10.15. *The morphism*

$$L^q_{4n}(\mathbb{Z}) \to \mathbb{Z} \qquad C \mapsto \tfrac{\mathrm{sgn}(H_{2n}(C)\otimes\mathbb{Q})}{8}$$

is a well-defined isomorphism.

Proof. It is clear that $L^q_{4n}(\mathbb{Z}) \to L^s_{4n}(\mathbb{Z}) \to \mathbb{Z}$ is injective (by the same argument as above) since hyperbolic forms are always quadratic. We now use the fact that the signature of each quadratic form is divisble by 8. For surjectiivity one uses the E_8-form which is an 8-dimensional, unimodular, even form of signature 8 given by

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

□

Proposition 10.16. *The morphism*

$$\mathrm{Arf} : L^q_{4n+2}(\mathbb{Z}) \to L^q_{4n+2}(\mathbb{F}_2) \to \mathbb{Z}/2 \qquad C \mapsto \mathrm{Arf}\left(H_{2n+1}(C/2)\right)$$

is an isomorphism

Proposition 10.17. *The morphism*

$$\mathrm{dR} : L^s_{4n+1}(\mathbb{Z}) \to \mathbb{Z}/2 \qquad C \mapsto \dim_{\mathbb{F}_2}\left(H_{2n}(C)[2]\right)$$

is an isomorphism.

Now we also want to compute the Grothendieck–Witt groups in this generality. Recall that for a ring R (possibly with involution) we defined the Grothendieck–Witt groups $\mathrm{GW}_0^s(R)$ and $\mathrm{GW}_0^q(R)$ as the group completion of the respective monoids of unimodular forms. Again if $\frac{1}{2} \in R$ then the two groups agree and we simply write $\mathrm{GW}_0(R)$.

Proposition 10.18. *For a field k of characteristic $\neq 2$ the Grothendieck–Witt $\mathrm{GW}_0(k)$ ring is generated by classes $[\alpha]$ for $\alpha \in k^\times$. It sits in a short exact sequence*

$$0 \to \mathbb{Z} \to \mathrm{GW}_0(k) \to W_0(k) \to 0$$

If k is quadratically closed then the map

$$\dim : \mathrm{GW}_0(k) \to \mathbb{Z}$$

is an isomorphism. [17]

Proof. The existence of orthogonal bases implies the first claim. For the second claim we observe that $L_1(k) = 0$ and $K_0(k)_{C_2} = \mathbb{Z}$. The last follows since in the qudratically closed case any 1-dimensional space is isomorphic to $(k, 1)$. $\square$

Example 10.19. We have that $\mathrm{GW}_0(\mathbb{C}, \mathrm{id}) = \mathbb{Z}$ via dimension and $\mathrm{GW}_0(\mathbb{C}, *) = \mathbb{Z} \oplus \mathbb{Z}$ via signature and dimension minus signature divided by 2.

Example 10.20. We have that $\mathrm{GW}_0^s(\mathbb{F}_2) = \mathbb{Z}$ given by dimension and $\mathrm{GW}_0^q(\mathbb{F}_2) = \mathbb{Z} \oplus \mathbb{Z}/2$ given by

$$(\tfrac{dim}{2}, \mathrm{Arf}) : \mathrm{GW}_0^q(\mathbb{F}_2) \to \mathbb{Z} \oplus \mathbb{Z}/2\ .$$

Example 10.21. The morphism

$$(\mathrm{sgn}, \frac{\dim - \mathrm{sgn}}{2}) : \mathrm{GW}_0(\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}$$

is an isomorphism.

11. Non-abelian derived functors

Assume that we are given a Poincaré-∞-category. Then we recall from Corollary 9.34 that for bounded weight and t-structures we can (at least in theory) compute the L-groups and compare them to classical invariants. In this section we explain how to modify a general quadratic functor Ϙ without changing the underlying duality to be of this specific type. The main result will be the following:

Theorem 11.1. *Assume that $\mathcal{C}$ admits a bounded weight structure. Then the forgetful functor*

$$\mathrm{Fun}^q(\mathcal{C}) \to \mathrm{Fun}^q(\mathcal{C}^{w\heartsuit})$$

is an equivalence of ∞-categories.

Here $\mathrm{Fun}^q(\mathcal{C})$ denotes quadratic functors in $\mathcal{C}$ and $\mathrm{Fun}^q(\mathcal{C}^{w\heartsuit})$ denotes quadratic functors

$$\left(\mathcal{C}^{w\heartsuit}\right)^{\mathrm{op}} \to \mathrm{Sp},$$

in the sense of Eilenberg–MacLane, i.e. reduced functors whose second cross effect is bilinear. We will deduce Theorem 11.1 from the following result:

Proposition 11.2. *Assume that $\mathcal{C}$ admits a bounded weight structure. Then the forgetful functors*

$$\mathrm{Fun}^{\mathrm{sif}}(\mathrm{Ind}(\mathcal{C}_{\geq 0})^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Fun}'(\mathcal{C}_{\geq 0}^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Fun}\left(\left(\mathcal{C}^{w\heartsuit}\right)^{\mathrm{op}}, \mathrm{Sp}\right)$$

are equivalences, where $\mathrm{Fun}^{\mathrm{sif}}$ *is the ∞-category that preserves sifted limits and* Fun' *is the ∞-category of functors that send pushouts to totalizations. The inverses are given by left Kan extension.*

In fact, $\mathrm{Ind}(\mathcal{C}_{\geq 0})$ is the universal ∞-category obtained from $\mathcal{C}^{w\heartsuit}$ by adjoining sifted colimits (or equivalently Quillen's non-abelian derived category) and that $\mathcal{C}_{\geq 0}$ is the universal ∞-category obtained by adjoining finite geometric realizations.

[17] In fact the converse is also true: if this map is an isomorphism, then k is quadratically closed.

Proof. We first note that the first functor is an equivalence. To see this we use that the forgetful functor

$$\mathrm{Fun}^{\mathrm{fil}}(\mathrm{Ind}(\mathcal{C}_{\geq 0})^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Fun}(\mathcal{C}_{\geq 0}^{\mathrm{op}}, \mathrm{Sp})$$

is an equivalence. Moreover a functor preserves sifted colimits precisely if it preserves finite geometric realizations and filtered colimits.

Now we want to show that the composite morphism is an equivalence. To this end we claim that $\mathrm{Ind}(\mathcal{C}_{\geq 0})$ is obtained from $\mathcal{C}^{w\heartsuit}$ by adjoining sifted colimits. By general theory (see HTT), since $\mathcal{C}^{w\heartsuit}$ already admits all coproducts, this is obtained by the universal ∞-category with all colimits that receives a coproduct preserving functor from $\mathcal{C}^{w\heartsuit}$, i.e. we have to show that

$$\mathrm{Ind}(\mathcal{C}_{\geq 0}) \simeq \mathrm{Fun}^{\Sigma}((\mathcal{C}^{w\heartsuit})^{\mathrm{op}}, \mathcal{S})$$

In general we have that

$$\mathrm{Ind}(\mathcal{C}_{\geq 0}) \simeq \mathrm{Fun}^{\mathrm{fin}}(\mathcal{C}_{\geq 0}^{\mathrm{op}}, \mathcal{S})$$

so that we have to argue that the restriction

$$\pi^* : \mathrm{Fun}^{\mathrm{fin}}(\mathcal{C}_{\geq 0}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}^{\Sigma}((\mathcal{C}^{w\heartsuit})^{\mathrm{op}}, \mathcal{S})$$

is an equivalence with inverse given by left Kan extension $\pi_!$. For this we note that left Kan extension indeed lands in the claim subcategory (exercise) and that by fully faithfulness the composite $\pi^* \circ \pi_!$ is the identity. It thus suffices to show that the canonical morphism $\pi_! \circ \pi^* \to \mathrm{id}$ is an equivalence. We reduced to showing that π^* detects equivalences. But this is clear since every object of $\mathcal{C}_{\geq 0}$ is a finite colimit of objects in $\mathcal{C}^{w\heartsuit}$. $\square$

Proof of Theorem 11.1. It follows immediately from the last Proposition and Proposition 3.14 that we get an equivalence between functor categories

$$\mathrm{Fun}^q(\mathcal{C}_{\geq 0}^{\mathrm{op}}) \simeq \mathrm{Fun}^q(\mathcal{C}^{w\heartsuit}) \ .$$

Thus everything that is left is to show that the forgetful functor

$$\mathrm{Fun}^q(\mathcal{C}) \to \mathrm{Fun}^q(\mathcal{C}_{\geq 0}^{\mathrm{op}})$$

is an equivalence. To this end we construct an explicit inverse given as the composition of left Kan extension followed by 2-excisive approximation... $\square$

Remark 11.3. For a given quadratic functor $Ϙ$ defined on the weighty heart we can give a rather explicit description of the associated functor $\overline{Ϙ} : \mathcal{C}_{\geq 0}^{\mathrm{op}} \to \mathrm{Sp}$. Namely one writes an object $X \in \mathcal{C}$ as a finite geometric realization of an objects in $\mathcal{C}^{w\heartsuit}$, i.e.

$$X = \mathrm{colim}_{\Delta^{\mathrm{op}}} X_i$$

Then $Ϙ(X) = \lim_{\Delta} Ϙ(X_i)$. This is the description of non-abelian derived functors given by Dold-Puppe.

Corollary 11.4. *For a given stable ∞-category $\mathcal{C}$ with a bounded weight structure there is an equivalence between compatible Poincaré structures on $\mathcal{C}$ whose duality preserves the weight heart and Poincaré structures on $\mathcal{C}^{w\heartsuit}$* [18]

[18]By this we shall simply mean a quadratic functor on $\mathcal{C}^{w\heartsuit}$ with values in spectra such that the second cross effect is connective and represented by a self-duality. Earlier we had required the whole functor to take values in connective spectra.

Recall from our surgery process that we would like our quadratic functor to take values in connective spectra when evaluated on the heart (or more generally on the weight-coconnective part).

Corollary 11.5. *For every quadratic functor* Ω *on* $\mathcal{C}$ *there are new quadratic functors* $\Omega' \to \Omega$ *and* $\Omega' \to \Omega''$ *such that when restricted to* $X \in \mathcal{C}^{w\heartsuit}$ *the map*

$$\Omega'(X) \to \Omega(X)$$

is a connective cover and $\Omega''(X) = \pi_0(\Omega)(X)$*. If the duality is compatible with the weight structure then* Ω' *is also Poincaré and if the weight heart is an ordinary category then* Ω'' *is also Poincaré.*

Now we specialize the previous discussion to the case of rings. Let R be a ring with involution. Then for any Poincaré structure

$$\Omega : \mathrm{Proj}_R^{\mathrm{op}} \to \mathrm{Ab}$$

we get an induced Poincaré structure

$$\overline{\Omega} : \mathcal{D}^{\mathrm{perf}}(R)^{\mathrm{op}} \to \mathrm{Sp}$$

such that

$$L_n(\mathcal{D}^{\mathrm{perf}}(R), \overline{\Omega}) = L_n(\mathcal{D}^{\mathrm{perf}}(R), \overline{\Omega}, w) \ .$$

Moreover we have that $\mathrm{GW}^{\Omega}(R) = \mathrm{GW}_0(\mathcal{D}^{\mathrm{perf}}(R), \overline{\Omega})$. If R is noetherian of finite global dimension we even get that those are isomorphic to

$$L_n(\mathcal{D}^{\mathrm{perf}}(R), \overline{\Omega}) = L_n(\mathcal{D}^{\mathrm{perf}}(R), \overline{\Omega}, w, \tau) = \frac{\{\text{Poincaré objects of lenght } k(n)\}}{\{\text{Those with Lagrangians of length } k(n)\}}$$

where

$$k(n) = \begin{cases} 2 & \text{for } n \leq 1 \\ n+1 & \text{for } 1 \leq n \leq d \\ d+1 & \text{for } n \geq d \end{cases}$$

If for example R is of dimension 1 then we simply get that those groups are isomorphic to the respective Witt groups in each degree.

Definition 11.6. *For any ring with involution* R *we define the genuine symmetric* $L^{gs}(R)$ *and genuine quadratic L-groups* $L^{gq}(R)$ *as*

$$L_n(\mathcal{D}^{\mathrm{perf}}(R), \overline{\Omega})$$

for Ω *the functor that assigns the abelian group of symmetric or quadratic forms, i.e.*

$$\Omega^{gs}(P) = \mathrm{Hom}_{\mathrm{Proj}_{R\otimes R}}(P \otimes P, R)^{C_2}$$
$$\Omega^{gq}(P) = \mathrm{Hom}_{\mathrm{Proj}_{R\otimes R}}(P \otimes P, R)_{C_2} \ .$$

We will also sometimes abbreviate 'gs' to 'g' and just speak of genuine L-theory.

We observe that there is a sequence of maps

$$L_*^q(R) \to L_*^{gq}(R) \to L_*^{gs}(R) \to L_*^s(R)$$

induced from the maps

$$B_{hC_2} \to B_{C_2} \xrightarrow{Nm} B^{C_2} \to B^{hC_2}$$

for B the abelian group $\mathrm{Hom}_{\mathrm{Proj}_{R\otimes R}}(P \otimes P, R)$ with the flip action. In the case that $\frac{1}{2} \in R$ then all these maps are equivalences. In particular for fields k of characteristic

$\neq 2$ we get that $L^{gq}_*(k) = L^{gs}_*(k)$ is given by the Witt group of k n degree $4n$ and 0 else.

Proposition 11.7. *The linear part of the genuine symmetric functor is given by the functor*

$$\mathcal{D}^{\mathrm{perf}}(R)^{\mathrm{op}} \to \mathrm{Sp} \qquad X \mapsto \mathrm{map}_R(X, \tau_{\geq 0} R^{tC_2})$$

The linear part of the genuine quadratic functor is given by the functor

$$\mathcal{D}^{\mathrm{perf}}(R)^{\mathrm{op}} \to \mathrm{Sp} \qquad X \mapsto \mathrm{map}_R(X, \tau_{\geq 2} R^{tC_2})$$

Proof. Establish the squares... □

Remark 11.8. There is also a functor in between, given by *even* forms.

We now also want to compute the genuine L-groups for other fields and rings (especially the integers). In general we have the following result:

Theorem 11.9. *Let R be noetherian with involution. For the canonical morphisms*

$$L^q_*(R) \to L^{gq}_*(R) \to L^{gs}_*(R) \to L^s_*(R)$$

we have that

(1) *the first morphism is an isomorphism in degrees $* \leq 1$ and surjective in degree $* = 2$.*
(2) *The second is an isomorphism in degrees $* \notin [-2, \dim +2]$, surjective in degree $* = -2$ and injective in degree $\dim +2$.*
(3) *The third is bijective in degrees $* \geq \dim -1$ and injective for $* = \dim -2$.*

For arbitrary rings (that is without the noetherian assumption) the lower estimates still hold. In general we have that $L^{gq}_*(R) = L^{gs}_{*+4}(R)$.

Proof. The first three estimates follow from our surgery results (with some care about comparing the functors) and will be skipped. For the last claim we have to show that we have an equivalence

$$(\mathcal{D}^{\mathrm{perf}}(R), Ϙ^{gq}) \xrightarrow{[-2]} (\mathcal{D}^{\mathrm{perf}}(R), Ϙ^{gs})$$

of Poincaré-∞-categories. To see this we have to analyse the genuine symmetric functor on $P[-2]$ for P a finitely generated, projective R-module. To do this we note that the bilinear part is given by symmetric even forms. □

Theorem 11.10. *We have that*

$$L^{gq}_*(\mathbb{F}_2) = \begin{cases} \mathbb{Z}/2 & \text{for } * = 2n, n \neq 1 \\ 0 & \text{else} \end{cases} \qquad L^{gs}_*(\mathbb{F}_2) = \begin{cases} \mathbb{Z}/2 & \text{for } * = 2n, n \neq -1 \\ 0 & \text{else} \end{cases}$$

and

$$L^{gq}_*(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 4n \\ \mathbb{Z}/2 & \text{for } * = 4n+1, n \geq 1 \\ \mathbb{Z}/2 & \text{for } * = -4n-2, n \geq 0 \\ 0 & \text{else} \end{cases} \qquad L^{gs}_*(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 4n \\ \mathbb{Z}/2 & \text{for } * = 4n+1, n \geq 0 \\ \mathbb{Z}/2 & \text{for } * = -4n-2, n \geq 1 \\ 0 & \text{else} \end{cases}$$

Proof. By the periodicity we can restrict attention to the genuine quadratic cases. For $\mathbb{F}_2$ we find that they agree with the quadratic ones in degrees ≤ 1 and with the symmetric ones in degrees ≥ 3. Thus in these cases the result follows from our

previous discussions. It only remains to determine the case $*=2$ in which we find that
$$\mathbb{Z}/2 = L^q_2(\mathbb{F}_2) \to L^{gq}_2(\mathbb{F}_2) \to L^s_2(\mathbb{F}_2) = \mathbb{Z}/2$$
is a epi-mono factorization. But the composite map is 0 by Proposition 10.8 so that the result also follows.

In the case of the integers the result (in the genuine quadratic case) follows in degrees ≤ 1 and ≥ 4 also directly from the previous computations. Thus it remains to determine the cases $*=2$ and $*=3$. In the first case we get an epimorphism
$$\mathbb{Z}/2 = L^q_2(\mathbb{Z}) \to L^{gq}_2(\mathbb{Z}) \ .$$
and in the second case a monomorphism
$$L^{gq}_3(\mathbb{Z}) \to L^s_3(\mathbb{Z}) = 0 \ .$$
In the $*=3$ case the result immediately follows and in the $*=2$ case one can show that the group is indeed zero, as it can be computed as the Witt group of even antisymmetric forms. □

12. Higher Grothendieck–Witt groups

We recall that for a ring R with involution we can define the spectra
$$\mathrm{GW}^s(R) = \{\text{groupoid of unimodular, symmetric forms over } R\}^{\mathrm{grp}}$$
$$\mathrm{GW}^q(R) = \{\text{groupoid of unimodular, quadratic forms over } R\}^{\mathrm{grp}} \ .$$
Here we take the group completion with respect to direct sum.

Construction 12.1. We hope the reader is familiar with the construction of connective spectra using group completion. Let us quickly sketch this, also to review the terminology... Digression about $\mathbb{E}_\infty$-spaces and conncective spectra, group completion etc?

More generally for an additive ∞-category $\mathcal{A}$ with a Poincaré-structure $\Game$ we define
$$\mathrm{GW}(\mathcal{A}, \Game) = \{\infty\text{-groupoid of } \Game\text{-Poincaré forms in } \mathcal{A}\}^{\mathrm{grp}} \ .$$
This is by definition a connective spectrum and has the property that its π_0 is given by our old $\mathrm{GW}_0(\mathcal{A}, \Game)$.

We now want to define the Grothendieck-Witt groups also for general Poincaré-∞-categories. The general idea is to introduce for any Poincaré-∞-category $(\mathcal{C}, \Game)$ a (connective) spectrum
$$\mathrm{GW}(\mathcal{C}, \Game) \in \mathrm{Sp}_{\geq 0}$$
which generalizes our previous definition of $\mathrm{GW}_0(\mathcal{C}, \Game)$ in that this is π_0 of our spectrum.

Definition 12.2 (Sketch)**.** *The connective Grothedieck–Witt spectrum*
$$\mathrm{GW}(\mathcal{C}, \Game) := \Omega|\,\mathrm{Cob}(\mathcal{C}, \Game)|$$
where $\mathrm{Cob}(\mathcal{C}, \Game)$ *is the cobodism ∞-category of $(\mathcal{C}, \Game)$ which is informally given as follows:*

Objects are (-1)-dimensional Poincaré objects in $(\mathcal{C}, \Game)$. A morphism (X, q) to (X', q') is given by a Lagrangian (aka nullbordism) of $(X \oplus X', q - q')$. One should think of the latter as a cobordism from (X, q) to (X', q') similar to the case of manifolds. This ∞-category is symmetric monoidal under direct sum so that $\Omega|\,\mathrm{Cob}(\mathcal{C}, \Game)|$ really becomes a connective spectrum.

Let's first take a step back and discuss the analogous K-theory versions. Recall that for an additive ∞-category $\mathcal{A}$ we have that

$$K(\mathcal{A}) = (A^{\simeq}, \oplus)^{\mathrm{grp}} \in \mathrm{Sp}_{\geq 0}$$

Definition 12.3. *The K-theory spectrum of a stable ∞-category $\mathcal{C}$ is defined as*

$$K(\mathcal{C}) := \Omega|\operatorname{Span}(\mathcal{C})|$$

where $\operatorname{Span}(\mathcal{C})$ *is the ∞-category of spans in $\mathcal{C}$, informally given as the ∞-category whose objects are the objects of $\mathcal{C}$ and whose morphisms are given by spans $X \leftarrow L \rightarrow Y$. A 2-morphism is given by an equivalence of spans. Composition of spans is given by pullback and $\Omega|\operatorname{Span}(\mathcal{C})|$ becomes a commutative group object under direct sum of spans.*

We will make this definition precise soon. In fact this is really Quillen's definiton by means of the Q-construction, though he did not do this in the setting of stable ∞-categories of course. In our generality this is (a special case of) a definition due to Barwick–Rognes. Before we give a precise definition of the ∞-category of spans let us understand why this definition generalizes the definition of $K_0(\mathcal{C})$ for a stable ∞-category $\mathcal{C}$.

Lemma 12.4. *The space $|\operatorname{Span}(\mathcal{C})|$ is connected and $\pi_0(K(\mathcal{C})) = \pi_1(|\operatorname{Span}(\mathcal{C})|)$ is naturally isomorphic to our old definition of $K_0(\mathcal{C})$.*

Proof. The first part is clear since for every object we have the span $A \leftarrow A \rightarrow 0$. The second part will be left as an exercise and we will present a nice conceptual solution soon. □

Definition 12.5. *Let $\mathcal{C}$ be a category. We define the* twisted arrow category *to be the category* $\operatorname{TwArr}\mathcal{C}$ *whose objects are arrows $f : A \to B$ in $\mathcal{C}$ and whose morphisms $f \to f'$ are commutative diagrams*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ A' \arrow[r, "f'"] & B' \arrow[u] \end{tikzcd} .$$

Composition is vertical stacking of diagrams. We let $\operatorname{TwArr}\Delta^n$ be the nerve of the twisted arrow category of the category $[n]$ whose nerve is Δ^n.

Example 12.6. For any poset P (like $[n]$) we get that $\operatorname{TwArr} P$ is given by the poset whose elements are pairs $i, j \in P$ with $i \leq j$ and with the relation that $ij \leq i'j'$ if $i \leq i'$ and $j \geq j'$.

We have that $\operatorname{TwArr}\Delta^0 = \Delta^0$ and $\operatorname{TwArr}\Delta^1$ can be depicted as

$$\begin{tikzcd} & 01 \arrow[dl] \arrow[dr] & \\ 00 & & 11 \end{tikzcd}$$

The category underlying $\mathrm{TwArr}\,\Delta^n$ can be depicted as follows:

$$\begin{tikzcd}
 & & & & 0n \arrow[dl] \arrow[dr] & & & & \\
 & & & \ldots \arrow[dl] \arrow[dr] & & \ldots \arrow[dl] \arrow[dr] & & & \\
 & & 02 \arrow[dl] \arrow[dr] & & 13 \arrow[dl] \arrow[dr] & & \ldots \arrow[dl] \arrow[dr] & & \\
 & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & \ldots \arrow[dl] \arrow[dr] & & n-1n \arrow[dl] \arrow[dr] & \\
00 & & 11 & & 22 & & \ldots & & nn
\end{tikzcd}$$

In particular we can think of a functor $\mathrm{TwArr}\,\Delta^n \to \mathcal{C}$ as a diagram in $\mathcal{C}$ of this shape.

Remark 12.7. The twisted arrow category is the (Cartesian) Grothendieck construction of the Hom-functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$. In fact one can define a twisted arrow category for any ∞-category, but we refrain from doing so.

Definition 12.8. *For any ∞-category $\mathcal{C}$ we define a functor $\mathcal{Q}_\bullet(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathrm{Cat}_\infty$ by letting*

$$\mathcal{Q}_n(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{TwArr}\,\Delta^n, \mathcal{C})$$

be the full subcategory consisting of those diagrams $F : \mathrm{TwArr}\,\Delta^n \to \mathcal{C}$ for which every square

$$\begin{tikzcd}
 & F(ii+2) \arrow[dl] \arrow[dr] & \\
F(ii+1) \arrow[dr] & & F(i+1i+2) \arrow[dl] \\
 & F(i+1i+1) &
\end{tikzcd}$$

with $0 \leq i \leq n-2$ is a pullback in $\mathcal{C}$. We let

$$\mathcal{Q}_\bullet^{\simeq}(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathcal{S}$$

be the associated functor obtained by discarding the non-invertible morphisms.

Proposition 12.9. *If $\mathcal{C}$ admits finite pullbacks then the object $\mathcal{Q}_\bullet^{\simeq}(\mathcal{C})$ is a complete Segal space, that is there is a unique ∞-category $\mathrm{Span}(\mathcal{C})$ such that we have a natural equivalence*

$$\mathcal{Q}_\bullet^{\simeq}(\mathcal{C}) \simeq \mathrm{Fun}(\Delta^\bullet, \mathrm{Span}(\mathcal{C}))^{\simeq} \ .$$

If $\mathcal{C}$ moreover admits a terminal object then $\mathrm{Span}(\mathcal{C})$ admits a symmetric monoidal struture given by

$$X \otimes Y = X \times Y.$$

and every object in $\mathrm{Span}(\mathcal{C})$ is self dual.

We warn the reader that the symmetric monoidal structure is not the Cartesian one, since $X \times Y$ is not the Cartesian product in the category $\mathrm{Span}(\mathcal{C})$.

Proof. One directly verifies the properties of a complete Segal space: the Segal condition follows since a diagram of the shape

$$\begin{tikzcd}
&&&& 0n \arrow[dl] \arrow[dr] &&&& \\
&&& \dots \arrow[dl] \arrow[dr] && \dots \arrow[dl] \arrow[dr] &&& \\
&& 02 \arrow[dl] \arrow[dr] && 13 \arrow[dl] \arrow[dr] && \dots \arrow[dl] \arrow[dr] && \\
& 01 \arrow[dl] \arrow[dr] && 12 \arrow[dl] \arrow[dr] && \dots \arrow[dl] \arrow[dr] && n-1n \arrow[dl] \arrow[dr] & \\
00 && 11 && 22 && \dots && nn
\end{tikzcd}$$

in $\mathcal{C}$ in which all squares are pullbacks is already determined by its restriction to

$$\begin{tikzcd}
& 01 \arrow[dl] \arrow[dr] && 12 \arrow[dl] \arrow[dr] && \dots \arrow[dl] \arrow[dr] && n-1n \arrow[dl] \arrow[dr] & \\
00 && 11 && 22 && \dots && nn
\end{tikzcd}$$

since the upper squares can be recovered by forming pullbacks (we skip the fine details). The completness basically follows since a morphism $A \leftarrow B \rightarrow C$ that is an equivalence as a span, is always induced by an equivalence $A \xrightarrow{\simeq} C$ in $\mathcal{C}$. Again we skip the details.

For the second part we need to equip $\mathrm{Span}(\mathcal{C})$ with the structure of a commutative monoid in Cat_∞ which is equivalent to equipping the functor $\mathcal{Q}_\bullet^{\simeq}(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathcal{S}$ with a refinement through $\mathrm{CMon}(\mathcal{S})$. For the latter we simply equip $\mathrm{Fun}(\mathrm{TwArr}\,\Delta^n, \mathcal{C})$ with the Cartesian structure and note that the induced functors from $[n] \to [m]$ in Δ preserves this structure.

Finally in order to verify that every object $X \in \mathcal{C}$ is self dual we simply exhibit evaluation and coevaluation of the duality as

$$\mathrm{ev} : X \times X \xleftarrow{\Delta} X \to \mathrm{pt}$$

$$\mathrm{coev} : \mathrm{pt} \leftarrow X \xrightarrow{\Delta} X \times X \ .$$

The Zig-Zag identity, then follows since the composite of spans

$$\begin{tikzcd}
& X \times X \arrow[dl, "p_1"'] \arrow[dr, "\mathrm{id}\times\Delta"] && X \times X \arrow[dl, "\Delta\times\mathrm{id}"'] \arrow[dr, "p_2"] & \\
X && X \times X \times X && X
\end{tikzcd}$$

is given by the identity span on X as one directly sees using that the pullback in the middle is equivalent to X. $\square$

Remark 12.10. Taking the dual morphism in $\mathrm{Span}(\mathcal{C})$ is very tautological as it just interchanges the orders of the legs of a span. Also the equivalence

$$\mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(A \times B, C) = \mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(A, B \times C)$$

induced from the abstract fact that $B^\vee = B$ is a tautology if one spells out what these spaces look like.

Now it is clear that any right exact functor $\mathcal{C} \to \mathcal{D}$ between ∞-categories with finite limits induces a symmetric monoidal functor

$$\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$$

by 'pointwise' application. In particular when applied to stable ∞-categories we get a functor from the ∞- category of stable ∞-categories to the ∞-category of symmetric monoidal ∞-categories. Then, as in Definition 12.3 above we geometrically realize this symmetric monoidal ∞-category to get a commutative monoid object

$$|\mathrm{Span}(\mathcal{C})| \in \mathrm{CMon}(\mathcal{S}) \ .$$

Taking Ω thus produces a commutative group

$$\Omega|\mathrm{Span}(\mathcal{C})| \in \mathrm{CGrp}(\mathcal{S}) \simeq \mathrm{Sp}_{\geq 0} \ .$$

Also note that taking Ω does not loose any information as the original space $|\mathrm{Span}(\mathcal{C})|$ was connected as shown in Lemma 12.4. Thus it was in fact already a group.

Now in the case of a ring R one could wonder about the relation between $K(R)$ defines as the group completion

$$K(R) = \{\text{groupoid of f.g. projective } R\text{-modules}\}^{\mathrm{grp}}$$

and

$$K\left(\mathcal{D}^{\mathrm{perf}}(R)\right) = \Omega|\mathrm{Span}(\mathcal{C})|$$

There is a canonical map

$$K(R) \to K\left(\mathcal{D}^{\mathrm{perf}}(R)\right) = \Omega|\mathrm{Span}(\mathcal{C})|$$

given by sending P to the span $0 \leftarrow P \to 0$.[19] Proposition 8.11 together with Lemma 12.4 shows that this morphism is an isomorphism on π_0. In fact we have the following more general result:

Theorem 12.11 (Gillet–Waldhausen, Fontes)**.** *Assume that a stable ∞-category $\mathcal{C}$ admits a bounded weight structures. Then the canonical map*

$$K(\mathcal{C}^{w\heartsuit}) \to K(\mathcal{C})$$

is an equivalence of connective spectra.

Proof. The idea is roughly the following: □

Now we want to turn our attention back to the case of a Poincaré-∞-category $(\mathcal{C}, \Omega)$ and define the corrsponding cobordism category $\mathrm{Cob}(\mathcal{C}, \Omega)$ explained in Definition 12.2. Roughly speaking this is a refinement of $\mathrm{Span}(\mathcal{C})$ where all the objects $X \in \mathrm{Span}(\mathcal{C})$ are equipped with the structure of Poincaré objects of dimension -1 and all the spans are equipped with the structure of algebraic cobordisms. Note that the funny dimension shift is such that the 'relevant' spans $0 \leftarrow X \to 0$ then encode Poincaré objects if dimension 0. The key to implement this idea technically is to take the $\mathcal{Q}$-construction and equip it with the structure of a Poincaré-∞-category.

[19]Formally this is implemented by considering a symmetric monoidal subcategory of $\mathrm{Span}(\mathcal{D}^{\mathrm{perf}}(R))$ with the single object 0 and the morphisms given by $0 \leftarrow P \to 0$ for P finitely generated projective. This then implements a deloop of the ∞-groupoid Proj_R.

Construction 12.12. Let $(\mathcal{C}, \Qoppa)$ be a Poincaré-∞-category. Then for any n we equip $\mathcal{Q}_n \subseteq \operatorname{Fun}(\operatorname{TwArr}\Delta^n, \mathcal{C})$ with a functor $\Qoppa_n$ given by

$$\Qoppa_n(F) = \lim_{(\operatorname{TwArr}\Delta^n)^{\mathrm{op}}} \Qoppa(F) \ .$$

We claim that this functor is in fact quadratic and even Poincaré. The first assertion immediately follows from the observation that the cross effect and the linear part are both given pointwise, i.e.

$$B_{\Qoppa_n}(F, G) = \lim_{(\operatorname{TwArr}\Delta^n)^{\mathrm{op}}} B_{\Qoppa}(F, G)$$
$$L_{\Qoppa_n}(F) = \lim_{(\operatorname{TwArr}\Delta^n)^{\mathrm{op}}} L_{\Qoppa}(F) \ .$$

These are clearly bilinear and linear. Now in order to see that $\Qoppa_n$ is Poincaré we have to identify the duality. First in the case $n = 0$ we have $\mathcal{C}$ and in the case $n = 1$ we find by an explicit calculation that

$$\begin{aligned}
&B_{\Qoppa_1}(A \leftarrow B \to C, A' \leftarrow B' \to C') \\
&= B_{\Qoppa}(A, A') \times_{B_{\Qoppa}(B,B')} B_{\Qoppa}(C, C') \\
&= \operatorname{map}_{\mathcal{C}}(A, DA') \times_{\operatorname{map}_{\mathcal{C}}(B,DB')} \operatorname{map}_{\mathcal{C}}(C, DC') \\
&= \operatorname{map}_{\mathcal{Q}_1}(A \leftarrow B \to C, DA' \leftarrow DA' \times_{DB'} DC' \to DC')
\end{aligned}$$

For the higher n it is then very easy to see that we in fact have that as pairs of a stable ∞-category together with a quadratic functor we have that

$$\mathcal{Q}_n = \mathcal{Q}_1 \times_{\mathcal{Q}_0} \dots \times_{\mathcal{Q}_0} \mathcal{Q}_1$$

meaning that the quadratic functors are also simply the pullbacks of the respective quadratic functors. As a result we get that all of the $\mathcal{Q}_n$ are also Poincaré where the duality is given by applying the respective dualities on the part

$$\begin{tikzcd}
& 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & \dots \arrow[dl] \arrow[dr] & & n-1n \arrow[dl] \arrow[dr] & \\
00 & & 11 & & 22 & & \dots & & nn
\end{tikzcd}$$

Now for any map $[n] \to [m]$ we get an induced transformation

$$(\mathcal{Q}_m, \Qoppa_m) \to (\mathcal{Q}_n, \Qoppa_n)$$

induced from the map $\operatorname{TwArr}\Delta^n \to \operatorname{TwArr}\Delta^m$ and the universal property of the limit. We claim that this is a Poincaré functor. This again simply follows by observing the effect on the duality of the relevant part of the diagram. In total we conclude that we have a refinement of $\mathcal{Q}_\bullet(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathrm{Cat}_\infty$ to a functor

$$\mathcal{Q}_\bullet(\mathcal{C}, \Qoppa) : \Delta^{\mathrm{op}} \to \mathrm{Cat}_\infty^p \ .$$

We can postcompose with the functor $\mathcal{P} : \mathrm{Cat}_\infty^p \to \mathcal{S}$ given by sending $(\mathcal{C}, \Qoppa)$ to the space $\mathcal{P}(\mathcal{C}, \Qoppa)$ of Poincaré objects. This way we obtain a functor

$$\mathcal{P}\mathcal{Q}_\bullet(\mathcal{C}, \Qoppa) : \Delta^{\mathrm{op}} \to \mathcal{S}$$

Proposition 12.13. *The object $\mathcal{P}\mathcal{Q}_\bullet(\mathcal{C}, \Qoppa)$ is a complete Segal space for any $(\mathcal{C}, \Qoppa)$. The associated ∞-category admits a canonical symmetric monoidal structure given by direct sum in which every object (X, q) is dualizable with dual $(X, -q)$.*

Proof. We argue as in the proof of Proposition 12.9. All the steps are verbatim the same, except for the last step we have to equip the unit and counit transformations with the respective forms. □

Definition 12.14. *For a given Poincaré-∞-category we let* $\mathrm{Cob}(\mathcal{C}, \Omega)$ *be the symmetric monoidal ∞-category with associated complete Segal space* $\mathcal{P}\mathcal{Q}_\bullet(\mathcal{C}, \Omega[1])$ *.*

Proof of Proposition 12.13. Similar... □

Lemma 12.15. *We have that* $|\mathrm{Cob}(\mathcal{C}, \Omega)|$ *is a group object in* $\mathcal{S}$ *and that its* π_0 *is isomorphic to* $L_{-1}(\mathcal{C}, \Omega)$ *and its* $\pi_1|\mathrm{Cob}(\mathcal{C}, \Omega)|$ *is naturally isomorphic to* $\mathrm{GW}_0(\mathcal{C}, \Omega)$*.*

Proof. It is clear that $\pi_0|\mathrm{Cob}(\mathcal{C}, \Omega)|$ is given by cobordism classes of -1-dimensional Poincaré objects. But this is precisely the definition of $L_{-1}(\mathcal{C}, \Omega)$, or at least equivalent to it (see Lemma 6.11 for details). From this we also see that $\pi_0|\mathrm{Cob}(\mathcal{C}, \Omega)|$ is a group and not just a monoid. In general a commutative monoid object in $\mathcal{S}$ is a group preicsely if π_0 is a group. We skip the presentation of π_1. □

Remark 12.16. As explained before we then define a connective spectrum as the spectrum associated with

$$\mathrm{GW}(\mathcal{C}, \Omega) := \Omega|\mathrm{Cob}(\mathcal{C}, \Omega)|$$

In contrast to the case of $\mathrm{Span}(\mathcal{C})$ we can in fact define a spectrum in $\mathrm{Sp}_{\geq -1}$ by considering the downshift of the spectrum associated with $|\mathrm{Cob}(\mathcal{C}, \Omega)|$. The connective cover agrees with our $\mathrm{GW}(\mathcal{C}, \Omega)$ and its (-1)st homotopy group agrees with $L_{-1}(\mathcal{C}, \Omega)$. One can in fact one can very naturally define a non-connective spectrum whose negative homotopy groups agree with the negative L-groups and whose connective cover is equivalent to our $\mathrm{GW}(\mathcal{C}, \Omega)$.

Example 12.17. We have that $\mathrm{GW}(\mathrm{Hyp}(\mathcal{C})) = K(\mathcal{C})$. To see this note that $\mathcal{P}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathcal{C}^{\simeq}$ and also observe that $\mathcal{Q}_n(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Hyp}(\mathcal{Q}_n(\mathcal{C}))$.

We now want to compare GW-theory of a ring R to the Grothendieck Witt theory of the associated perfect derived ∞-category. This can be done in greater generality, as in the case of K-theory.

Theorem 12.18 (Hebestreit-Steimle)**.** *Let* $(\mathcal{C}, \Omega)$ *be a Poincaré-∞-category with a bounded weight structure. Then the canonical map*

$$\mathrm{GW}\left(\mathcal{C}^{w\heartsuit}, \Omega|_{\mathcal{C}^{w\heartsuit}}\right) \to \mathrm{GW}(\mathcal{C}, \Omega)\ .$$

is an equivalence of connective spectra.

Corollary 12.19. *For any ring R with involution we have equivalences*

$$\mathrm{GW}^s(R) \simeq \mathrm{GW}(\mathcal{D}^{\mathrm{perf}}(R), \Omega^{gs}) \quad \text{and} \quad \mathrm{GW}^q(R) \simeq \mathrm{GW}(\mathcal{D}^{\mathrm{perf}}(R), \Omega^{gq})$$

More generally for any Poincaé functor

$$\Omega : \mathrm{Proj}_R^{\mathrm{op}} \to \mathrm{Ab}$$

we have an equivalence

$$\mathrm{GW}^{\Omega}(R) \simeq \mathrm{GW}(\mathcal{D}^{\mathrm{perf}}(R), \overline{\Omega})$$

where $\overline{\Omega}$ *is the non-abelian derived quadratic functor.*

13. Poincaré-Verdier-Sequences

In this section we want to discuss the arguable most important property of the Grothendieck–Witt spectrum. Let us warm up first by considering the respective property for K-theory.

Definition 13.1. *An exact functor $p : \mathcal{D} \to \mathcal{C}$ between stable ∞-categories is called a Verdier projection of $\mathcal{C}$ is obtained from $\mathcal{D}$ by Dwyer–Kan inverting a class W of weak equivalences. It is said to be left/right split if it admits a fully faithful left/right adjoint. It is said to be split or to exhibit a* recollement *of $\mathcal{D}$ if it admits both adjoints and both are fully faithful.*

Note that if p admits a fully faithful adjoint on either side then it is automatically a DK localization.

Example 13.2.
1. The projection $p : \mathcal{C} \oplus \mathcal{D} \to \mathcal{D}$ for every pair $\mathcal{C}, \mathcal{D}$ of stable ∞-categories is a split Verdier projection.
2. For every stable ∞-category $\mathcal{C}$ the target projection $\mathcal{C}^{\Delta^1} \to \mathcal{C}$ is a split Verdier projection.
3. The functor $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}) \to \mathcal{D}^{\mathrm{perf}}(\mathbb{Q})$ is a Verdier projection but does not admit either adjoint. To see this observe that any exact functor $\mathcal{D}^{\mathrm{perf}}(\mathbb{Q} \to \mathcal{D}^{\mathrm{perf}}(\mathbb{Z})$ is necessarily zero, since no object in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z})$ is rational.

Theorem 13.3. *(Additivity) Assume that we have a pullback square*

$$\begin{tikzcd} \mathcal{D}' \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathcal{C}' \arrow[r] & \mathcal{C} \end{tikzcd}$$

in which the vertical maps are split Verdier projections. Then the induced diagram on K-theory spectra is also a pullback.

Proof. The induced map $\mathrm{Span}(\mathcal{D}) \to \mathrm{Span}(\mathcal{C})$ is a Cartesian and coCartesian fibration as one can verify directly. Thus it is a realization fibration by a result of Steimle. Moreover we have that the induced square

$$\begin{tikzcd} \mathrm{Span}(\mathcal{D}') \arrow[r] \arrow[d] & \mathrm{Span}(\mathcal{D}) \arrow[d] \\ \mathrm{Span}(\mathcal{C}') \arrow[r] & \mathrm{Span}(\mathcal{C}) \end{tikzcd}$$

is a pullback since this is levelwise true for the $\mathcal{Q}$-construction. Together these facts show that the resulting square is a pullback of connective spectra. But since the map $\mathcal{D} \to \mathcal{C}$ admits a section it also follows that it is surjective on K_0 and thus we have a pullback of spectra (which happen to be connective). □

Example 13.4. We have that $K(\mathcal{C}^{\Delta^1}) = K(\mathcal{C}) \oplus K(\mathcal{C})$. More generally for any split Verdier projection $p : \mathcal{D} \to \mathcal{C}$ we have that $K(\mathcal{D}) = K(\mathcal{C}) \oplus K(\ker p)$. In particular this also implies that $K(\mathcal{D} \oplus \mathcal{C}) = K(\mathcal{D}) \oplus K(\mathcal{C})$. The last fact is of course also easy to check directly.

Remark 13.5.
- The analogous result to Theorem 13.3 for Verdier sequences is also true but harder to prove and we will not need it here.

- One can also show that K-theory as a functor from stable ∞-categories to spectra is the universal functor with a chosen map $\mathbb{S} \to K(\mathrm{Sp}^{\mathrm{fin}})$ and that has the property of Theorem 13.3. More precisely for any functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ together with a map $x : \mathbb{S} \to F(\mathrm{Sp}^{\mathrm{fin}})$ which satisfies the conclusion of the additivity Theorem there exists an essentially unique transformation $K \to F$ that carries the class of the sphere to x. This is a result of Barwick and Blumberg–Gepner–Tabuada.

Definition 13.6. *We say that a map*

$$(p, \eta) : (\mathcal{D}, \Qoppa) \to (\mathcal{C}, \Qoppa')$$

is a Poincaŕ-Verdier projection if the underlying functor is a Verdier projection and the transformation $\eta : \Qoppa \to p^*\Qoppa'$ exhibits $\Qoppa'$ as a left Kan extension of $\Qoppa$.

In the case of a Poincaŕ-Verdier projection we get an induced Poincaré-structure on $\ker(p)$ be restriction of $\Qoppa$. Then we will also refer to the sequence

$$(\ker p, \Qoppa|_{\ker p}) \to (\mathcal{D}, \Qoppa) \to (\mathcal{C}, \Qoppa')$$

as a Poincaré-Verdier-sequence.

Lemma 13.7. *For a map $(\mathcal{D}, \Qoppa) \to (\mathcal{C}, \Qoppa')$ of Poincaré-∞-categories the following are equivalent:*

(1) *The underlying map $p : \mathcal{D} \to \mathcal{C}$ admits a fully faithful left adjoint.*
(2) *The underlying map $p : \mathcal{D} \to \mathcal{C}$ admits a fully faithful right adjoint.*
(3) *The underlying map $p : \mathcal{D} \to \mathcal{C}$ exhibits $\mathcal{D}$ as a recollement.*

Proof. Assume that (1) holds with left adjoint L. Then the fully faithful functor L^{op} is right adjoint to $p^{\mathrm{op}} : \mathcal{D} \to \mathcal{C}$. But the dualities induced a commutative square

$$\begin{tikzcd} \mathcal{D} \arrow[r, "p"] \arrow[d, "D"] & \mathcal{C} \arrow[d, "D"] \\ \mathcal{D}^{\mathrm{op}} \arrow[r, "p^{\mathrm{op}}"] & \mathcal{C}^{\mathrm{op}} \end{tikzcd}$$

so that $p \simeq p^{\mathrm{op}}$ which shows that p has a fullly faithful right adjoint given by $X \mapsto L(X^{\mathrm{op}})^{\mathrm{op}}$. Thus (3) holds. The other implications are either trivial or work the same. $\square$

Definition 13.8. *In either of the cases of Lemma 13.7 we say that the map $(\mathcal{D}, \Qoppa) \to (\mathcal{C}, \Qoppa')$ exhibits $(\mathcal{D}, \Qoppa)$ as a Poincaré recollement or that p is a split Verdier projection.*

Example 13.9. For every Poincaré-∞-category $(\mathcal{C}, \Qoppa)$ the map $\mathrm{Met}(\mathcal{C}, \Qoppa) \to (\mathcal{C}, \Qoppa)$ is a split Poincaré-Verdier projection. Since we already know that the underlying map, given by the target projection $\mathcal{C}^{\Delta^1} \to \mathcal{C}$, is a split Verdier projection we only have to verify that $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is the left Kan extension of $\Qoppa_{\mathrm{met}} : (\mathcal{C}^{\Delta^1})^{\mathrm{op}} \to \mathrm{Sp}$ along the target projection. But the left Kan extension Kan be computed as the pullback along the left adjoint to the target projection (since there is an op and the contravariance). The left adjoint to the target projection is given by the functor

$$\mathcal{C} \to \mathcal{C}^{\Delta^1} \qquad X \mapsto (0 \to X) \ .$$

As a result the left Kan extension is given by

$$(p_!\Qoppa_{\mathrm{met}})(X) = \Qoppa_{\mathrm{met}}(0 \to X) = \mathrm{fib}(\Qoppa(X) \to \Qoppa(0)) = \Qoppa(X) \ .$$

The kernel is given by $(\mathcal{C}, Ϙ[-1])$ so that we have a split Poincaré–Verdier sequence

$$(\mathcal{C}, Ϙ[-1]) \to \mathrm{Met}(\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ)$$

Example 13.10. For any pair of Poincaré-∞-categories $(\mathcal{C}, Ϙ)$ and $(\mathcal{D}, Ϙ')$ the projection $(\mathcal{C} \oplus \mathcal{D}, Ϙ \oplus Ϙ') \to (\mathcal{C}, Ϙ')$ is a split Poincaré-Verdier projection. Again the only new claim is that $Ϙ'$ is the left Kan extension of $Ϙ \oplus Ϙ'$ along the projection, which is straighforward to verify.

Remark 13.11. The projection $\mathrm{Met}(\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ)$ is the universal split Poincaré–Verdier projection with kernel $(\mathcal{C}, Ϙ[-1])$ in the following sense: for any split Poincaré–Verdier sequence $p : (\mathcal{D}, Ϙ') \to (\mathcal{E}, Ϙ'')$ with kernel $(\ker p, Ϙ') = (\mathcal{C}, Ϙ[-1])$ there exists a pullback diagram

$$\begin{tikzcd}
(\mathcal{D}, Ϙ') \arrow[r] \arrow[d] & \mathrm{Met}(\mathcal{C}, Ϙ) \arrow[d] \\
(\mathcal{E}, Ϙ'') \arrow[r] & (\mathcal{C}, Ϙ)
\end{tikzcd} .$$

Theorem 13.12. *(Poincaré-Additivity) Assume that we have a pullback square of Poincaré-∞-categories*

$$\begin{tikzcd}
(\mathcal{D}', Ϙ_{\mathcal{D}'}) \arrow[r] \arrow[d] & (\mathcal{D}, Ϙ_{\mathcal{D}}) \arrow[d] \\
(\mathcal{C}', Ϙ_{\mathcal{C}'}) \arrow[r] & (\mathcal{C}, Ϙ_{\mathcal{C}})
\end{tikzcd}$$

in which the vertical maps are split Poincaré–Verdier projections. Then the induced diagram on GW-theories

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{D}', Ϙ_{\mathcal{D}'}) \arrow[r] \arrow[d] & \mathrm{GW}(\mathcal{D}, Ϙ_{\mathcal{D}}) \arrow[d] \\
\mathrm{GW}(\mathcal{C}', Ϙ_{\mathcal{C}'}) \arrow[r] & \mathrm{GW}(\mathcal{C}, Ϙ_{\mathcal{C}})
\end{tikzcd}$$

is also a pullback of connective *spectra.*

Proof. The proof works precisely as the one of Theorem 13.3: one shows that the induced functor

$$\mathrm{Cob}(\mathcal{D}, Ϙ_{\mathcal{D}}) \to \mathrm{Cob}(\mathcal{C}, Ϙ_{\mathcal{C}})$$

is a Cartesian and coCartesian fibration. Thus it is a realization fibration and the result follows by the observation that the square

$$\begin{tikzcd}
\mathrm{Cob}(\mathcal{D}', Ϙ_{\mathcal{D}'}) \arrow[r] \arrow[d] & \mathrm{Cob}(\mathcal{D}, Ϙ_{\mathcal{D}}) \arrow[d] \\
\mathrm{Cob}(\mathcal{C}', Ϙ_{\mathcal{C}'}) \arrow[r] & \mathrm{Cob}(\mathcal{C}, Ϙ_{\mathcal{C}})
\end{tikzcd}$$

is a pullback of ∞-categories which follows directly from the definition using the hermitian $\mathcal{Q}$-construction. Finally we use that taking loop spaces preserves pullbacks. $\square$

Corollary 13.13. *For any Poincaré–Verdier seuqence*

$$(\mathcal{E}, Ϙ_{\mathcal{E}}) \to (\mathcal{D}, Ϙ_{\mathcal{D}}) \to (\mathcal{C}, Ϙ_{\mathcal{C}})$$

there is a natural long exact sequence

$$\ldots \to \mathrm{GW}_1(\mathcal{D}, \Omega_{\mathcal{D}}) \to \mathrm{GW}_1(\mathcal{C}, \Omega_{\mathcal{C}}) \to \mathrm{GW}_0(\mathcal{E}, \Omega_{\mathcal{E}}) \to \mathrm{GW}_0(\mathcal{D}, \Omega_{\mathcal{D}}) \to \mathrm{GW}_0(\mathcal{C}, \Omega_{\mathcal{C}}) \ .$$

Remark 13.14. We warn the reader that in general the pullback is not a pullback of spectra as the map

$$\mathrm{GW}_0(\mathcal{D}, \Omega_{\mathcal{D}}) \to \mathrm{GW}_0(\mathcal{C}, \Omega_{\mathcal{C}})$$

is in general not surjective. In fact the proof of Theorem 13.12 shows that one can continue the sequence of Corollary 13.13 to the right as

$$\mathrm{GW}_0(\mathcal{D}, \Omega_{\mathcal{D}}) \to \mathrm{GW}_0(\mathcal{C}, \Omega_{\mathcal{C}}) \to L_{-1}(\mathcal{E}, \Omega_{\mathcal{E}}) \to L_{-1}(\mathcal{D}, \Omega_{\mathcal{D}}) \to L_{-1}(\mathcal{C}, \Omega_{\mathcal{C}})$$

With some more care one can even further continue with the lower L-groups. In other words: if we define non-connective spectrum $\mathrm{GW}(\mathcal{C}, \Omega)$ as indicated in Remark 12.16 then one does get a pullback of spectrum in Theorem 13.12.

Remark 13.15.
- One also has a version of Theorem 13.12 and Corollary 13.13 for Poincaré-Verdier projections that are not necessarily split, but we will not need this here.
- In fact, Grothendieck–Witt theory also enjoys a universal property similar to the one of K-theory discussed in Remark 13.5.

Recall that for any Poincaré-∞-category $(\mathcal{C}, \Omega)$ we have the map

$$\mathrm{Hyp}(\mathcal{C}) \to \mathrm{Met}(\mathcal{C}) \qquad (X, Y) \mapsto (X \to X \oplus DY)$$

of Poincaré-∞-categories as discussed in Example 5.18.

Corollary 13.16. *For any Poincaré-∞-category the induced map*

$$K(\mathcal{C}) \simeq \mathrm{GW}(\mathrm{Hyp}(\mathcal{C})) \to \mathrm{GW}(\mathrm{Met}(\mathcal{C}))$$

is an equivalence.

Proof. Both sit in fibre sequences... □

One should see the last statement as a categorification of the defining relation for GW_0 that metabolic and hyperbolic forms agree.

14. The fundamental long exact sequence

In this section we want to prove that there is a long exact sequence

$$\begin{tikzcd}
 & \ldots \arrow[r] & L_2(\mathcal{C}.\Omega) \arrow[dll] \\
H_1(C_2; K(R)) \arrow[r] & \mathrm{GW}_1(\mathcal{C}, \Omega) \arrow[r] & L_1(\mathcal{C}, \Omega) \arrow[dll] \\
H_0(C_2; K(R)) \arrow[r] & \mathrm{GW}_0(\mathcal{C}, \Omega) \arrow[r] & L_0(\mathcal{C}, \Omega \arrow[r] & 0
\end{tikzcd}$$

extending the one of Proposition 6.12 In order to establish this we first need to construct the C_2-action on the K-theory spectrum induced by the duality on $\mathcal{C}$ and the hyperbolic map.

Proposition 14.1. *For any Poincaré-∞-category $(\mathcal{C}, \Omega)$ the space $\mathcal{C}^{\simeq}$ carries a natural C_2-action induced by the duality D and there is a map*

$$(\mathcal{C}^{\simeq})_{hC_2} \to \mathcal{P}(\mathcal{C}, \Omega)$$

induced by sending X to the hyperbolic form on X.

Proof. Let B_{Ω} as usual denote the cross effect of Ω. We consider the Poincaré-∞-category

$$(\mathcal{C} \times \mathcal{C}, B_{\Omega})$$

with duality given by $(X, Y) \mapsto (DY, DX)$. It carries a C_2-action given by flipping the coordinates which refines to a map of Poincaré-∞-categories using the fact that B_{Ω} is a symmetric bilinear functor. Moreover there is a C_2-equivariant functor

$$(\mathcal{C} \times \mathcal{C}, B_{\Omega}) \to (\mathcal{C}, \Omega) \qquad (X, Y) \mapsto X \oplus Y$$

where we use that the map $(B_{\Omega}^{\Delta})_{hC_2}$. As a result of the functoriality of $\mathcal{P}$ we get a C_2-action on $\mathcal{P}(\mathcal{C} \times \mathcal{C}, B_{\Omega})$ and an induced map

$$\mathcal{P}(\mathcal{C} \times \mathcal{C}, B_{\Omega})_{hC_2} \to \mathcal{P}(\mathcal{C}, \Omega)\ .$$

Finally the claim now follows from the observation that

$$\mathcal{P}((\mathcal{C} \times \mathcal{C}, B_{\Omega}) = \mathcal{P}(\mathrm{Hyp}(\mathcal{C})) = \mathcal{C}^{\simeq}$$

which can be checked easily (see 5.11 for the π_0 statement). □

Proposition 14.2. *For any Poincaré-∞-category $(\mathcal{C}, \Omega)$ there is a C_2-action on* $\mathrm{Span}(\mathcal{C})$ *and consequently on $K(\mathcal{C})$ and an induced functor*

$$\mathrm{Span}(\mathcal{C})_{hC_2} \to \mathrm{Cob}(\mathcal{C}, \Omega)$$

induced by forming hyperbolic objects which induces a map of connective spectra

$$K(\mathcal{C})_{hC_2} \to \mathrm{GW}(\mathcal{C}, \Omega)\ .$$

Proof. Apply the last claim levelwise to the $\mathcal{Q}$-construction. □

Lemma 14.3. *For the hyperbolic category $(\mathcal{C}, \Omega) = (\mathcal{D} \times \mathcal{D}^{\mathrm{op}}, \mathrm{map}_{\mathcal{D}})$ the maps*

$$(\mathcal{C}^{\simeq})_{hC_2} \to \mathcal{P}(\mathcal{C}, \Omega) \qquad \textit{and} \qquad K(\mathcal{C})_{hC_2} \to \mathrm{GW}(\mathcal{C}, \Omega)\ .$$

are equivalences.

Proof. It suffices to verify the first claim, the second follows by forming the realization. But in the case of the hyperbolic category we find that $\mathcal{P}(\mathcal{C}, \Omega) = \mathcal{D}^{\simeq}$ and that $\mathcal{C}^{\simeq} = \mathcal{D}^{\simeq} \times \mathcal{D}^{\simeq}$. Unfolding the constructions we get that the C_2-action flips the factors and thus the claim follows. □

Definition 14.4. *For any Poincaré-∞-category $(\mathcal{C}, \Omega)$ we define a connective spectrum*

$$l(\mathcal{C}, \Omega) := \mathrm{cof}(K(\mathcal{C})_{hC_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, \Omega))$$

Theorem 14.5. *The homotopy groups of $l(\mathcal{C}, \Omega)$ are naturally isomorphic to the L-groups.*

Proof. We first proof that there is a natural isomorphism

$$L_0(\mathcal{C}, \Omega) \cong \pi_0(l(\mathcal{C}, \Omega))\ .$$

To this end we simply observe that by definition we have an exact sequence

$$\pi_0\left(K(\mathcal{C})_{hC_2}\right) \to \pi_0(\mathrm{GW}(\mathcal{C}, \Omega)) \to \pi_0(l(\mathcal{C}, \Omega)) \to 0$$

that exhibits $\pi_0(l(\mathcal{C}, \Omega))$ as the cokernel of the π_0-effect of the hyperbolic map. But this is the hyperbolic map discussed earlier so that the π_0-claim follows from Proposition 6.12. Now we oberserve that for any given split Poincaré-Verdier sequence

$$(\mathcal{E}, \Omega_E) \to (\mathcal{D}, \Omega_D) \to (\mathcal{C}, \Omega_C)$$

we get a map of induced sequences

$$\begin{tikzcd}
K(\mathcal{E})_{hC_2} \arrow[r] \arrow[d] & K(\mathcal{D})_{hC_2} \arrow[r] \arrow[d] & K(\mathcal{C})_{hC_2} \arrow[d] \\
\mathrm{GW}(\mathcal{E}, \Qoppa) \arrow[r] \arrow[d] & \mathrm{GW}(\mathcal{D}, \Qoppa) \arrow[r] \arrow[d] & \mathrm{GW}(\mathcal{C}, \Qoppa) \arrow[d] \\
l(\mathcal{E}, \Qoppa) \arrow[r] & l(\mathcal{D}, \Qoppa) \arrow[r] & l(\mathcal{E}, \Qoppa)
\end{tikzcd}$$

of which the first two horizontal sequences are fibre sequences by Additivity (Theorem 13.3) and Poincaré-Additivity (Theorem 13.12). We deduce that the lower sequence is a fibre sequence except for a potential π_0-issue. But this immediately gets resolved by the ..

Now we apply this observation to the universal split Poincaré–Verdier sequence to get a long exact sequence

$$... \to \pi_n l(\mathrm{Met}(\mathcal{C}, \Qoppa) \to \pi_n l(\mathcal{C}, \Qoppa) \to \pi_{n-1} l(\mathcal{C}, \Qoppa[-1]) \to \pi_{n-1}(l(\mathrm{Met}(\mathcal{C}, \Qoppa)) \to ...$$

Now combining Corollary 13.16 and Lemma 14.3 we find that $l(\mathrm{Met}(\mathcal{C}, \Qoppa) = 0$. As a result we get an isomorphism

$$\pi_n l(\mathcal{C}, \Qoppa) = \pi_{n-1}(l(\mathcal{C}, \Qoppa[-1]) = ... = \pi_0(l(\mathcal{C}, \Qoppa[-n])$$

But by the first part of the proof we have

$$\pi_0(\mathrm{Met}(\mathcal{C}, \Qoppa[-n]) = L_0(\mathcal{C}, \Qoppa[-n]) = L_n(\mathcal{C}, \Qoppa)$$

which finishes the proof. □

Remark 14.6. Again as before, if one uses non-connective versions of GW then the cofibre of the map

$$K(\mathcal{C})_{hC_2} \to \mathrm{GW}(\mathcal{C}, \Qoppa)$$

will be a non-connective spectrum whose homotopy groups are the L-groups in all degrees.

There is also a more canonical way to define a spectrum $L(\mathcal{C}, \Qoppa)$ whose homotopy groups are the L-groups and which has been worked out by Lurie in []. Then one can promote Theorem 14.5 to an equivalence

$$l(\mathcal{C}, \Qoppa) \simeq \tau_{\geq 0} L(\mathcal{C}, \Qoppa) \ .$$

of spectra.

Finally we note that we have shown in the proof of Theorem 14.5 that l vanishes on metabolic Poincaré-∞-categories and that it satisfies Poincaré-Additivity. These two facts together are equivalent to

We define for a ring R with involution spectra $l^{gs}(R)$ and $l^{gq}(R)$ and note that they habe homotopy groups given by the classical L-groups by Definition 7.19.

Corollary 14.7. *For any ring R with involution we get fibre sequences of spectra*

$$K(R)_{hC_2} \to \mathrm{GW}^s(R) \to l^{gs}(R) \qquad \text{and} \qquad K(R)_{hC_2} \to \mathrm{GW}^q(R) \to l^{gq}(R)$$

Corollary 14.8. *We have equivalences*

$$\mathrm{GW}_*(\mathcal{C}, \Qoppa)[\tfrac{1}{2}] \simeq K_*(\mathcal{C})[\tfrac{1}{2}]^{C_2} \oplus L_*(\mathcal{C}, \Qoppa)[\tfrac{1}{2}]$$

for all $* \geq 0$.

Proof. After inverting 2 the fibre sequence

$$K(\mathcal{C})_{hC_2} \to \mathrm{GW}(\mathcal{C}, \Omega) \to l(\mathcal{C}, \Omega)$$

splits by the map

$$\mathrm{GW}(\mathcal{C}, \Omega)[\tfrac{1}{2}] \to \mathrm{GW}(\mathcal{C})^{hC_2}[\tfrac{1}{2}] \to \big(\mathrm{GW}(\mathcal{C}, \Omega)[\tfrac{1}{2}]\big)^{hC_2} \xrightarrow{\simeq} K(\mathcal{C})[\tfrac{1}{2}]_{hC_2} = K(\mathcal{C})_{hC_2}[\tfrac{1}{2}]$$

which shows the claim. $\square$

15. GW-theory of the integers

Now in this section we want to apply our fundamental long exact sequence to the case of the intergers to compute the Grothendieck–Witt groups of the integers.

Let us first sketch the computation away from the prime 2, which was (essentially) known before. We have for $* \geq 0$ an isomorphism

$$\mathrm{GW}^s_*(\mathbb{Z})[\tfrac{1}{2}] = L^{gs}_*(\mathbb{Z})[\tfrac{1}{2}] \oplus \big(K_*(\mathbb{Z})[\tfrac{1}{2}]\big)^{C_2} = \begin{cases} \mathbb{Z}[\frac{1}{2}] \oplus \big(K_*(\mathbb{Z})[\frac{1}{2}]\big)^{C_2} & * = 0 \mod 4 \\ \big(K_*(\mathbb{Z})[\frac{1}{2}]\big)^{C_2} & \text{else} \end{cases}$$

Thus it remains to work out what the C_2-action does on the K-groups $K(\mathbb{Z})[\frac{1}{2}]$. To this end we use the Quillen-Lichtenbaum conjecture, which implies that one has for ℓ an odd prime isomorphisms

$$K_{2n-2}(\mathbb{Z})_\ell = H^2_{\text{proét}}\left(\mathbb{Z}[\tfrac{1}{\ell}]; \mathbb{Z}_\ell(n)\right)$$
$$K_{2n-1}(\mathbb{Z})_\ell = H^1_{\text{proét}}\left(\mathbb{Z}[\tfrac{1}{\ell}]; \mathbb{Z}_\ell(n)\right)$$

where $\mathbb{Z}_\ell(n) = K_{2n}(-)_\ell$ is the Tate twist. The involution entirely acts on the sheaf $\mathbb{Z}_\ell(n)$ by naturality. We want to compute what the involution does on on points, i.e. on $K_{2n}(k)_\ell$ for k a strict Henselian local ring. But by results of Gabber and Suslin these are (canonically enough) equivalent to $K_{2n}(k)_\ell = \pi_{2n}(ku_\ell)$ so that it suffices to work out what the involution on ku induced by the trivial involution $\mathbb{C} \to \mathbb{C}$ does on homotopy. This involution induces the complex conjugation on the K-theory spectrum ku (also equivalent to the Adams operation ψ^{-1}) so that it sends the Bott element β to $-\beta$. As a result the involution acts as $(-1)^n$ on β^n and thus also by $(-1)^n$ on $\mathbb{Z}_\ell(n)$. As a consequence we find that the involution acts by multiplication with $(-1)^n$ on $K_{2n-2}(\mathbb{Z})_\ell$ and $K_{2n-1}(\mathbb{Z})_\ell$ for ℓ odd. Thus by finite generation it also acts by $(-1)^n$ on $K_{2n-2}(\mathbb{Z})[\frac{1}{2}]$ and $K_{2n-1}(\mathbb{Z})[\frac{1}{2}]$ and we obtain the following result:

Theorem 15.1. *We have for $* \geq 0$ isomorphisms*

$$\mathrm{GW}^s_*(\mathbb{Z})[\tfrac{1}{2}] = \begin{cases} \mathbb{Z}[\frac{1}{2}] & * = 0 \mod 4 \\ 0 & * = 1 \mod 4 \\ K_*(\mathbb{Z})[\frac{1}{2}] & * = 2,3 \mod 4\ . \end{cases}$$

The order of the groups $K_*(\mathbb{Z})[\frac{1}{2}]$ that appear here, which are by the above discusssion given by étale cohomology groups, is known in all degrees and can be expressed in terms of denominators of Bernoulli numbers. For $* = 3 \mod 4$ these groups are known for be cyclic in all degrees and for $* = 2 \mod 4$ they are known to be cyclic for $* \leq 20000$ and conjectured to be cyclic in all degrees.

Now we want to understand the Grothendieck–Witt groups at the prime 2. To this extend we shall use a model for K-theory of the integers conjectured first by

Bökstedt and then shown by Dwyer-Friedlander to be implied by the 2-primary Lichtenbaum-Quillen conjecture which was eventually resolved by Voevodsky:

Theorem 15.2. *There is a fibre sequence*

$$K(\mathbb{Z})_2 \to \mathrm{ko}_2 \xrightarrow{c(\psi^3-1)} \tau_{\geq 4}\mathrm{ku}$$

of spectra where $(-)_2$ *denotes 2-adic completion, the first map is the canonical map induced from* $\mathbb{Z} \to \mathbb{R}$ *and the second map is the composite of the Adams operation* ψ^3 *minus the identity*

$$\mathrm{ko}_2 \xrightarrow{\psi^3-1} \tau_{\geq 4}\mathrm{ko}_2 \tag{6}$$

and the 4*-connective cover of the complexification map* $c : \mathrm{ko} \to \mathrm{ku}$*.*

We of course have that $\tau_{\geq 4}\mathrm{ku} = \mathrm{bsu} \simeq \mathrm{ku}[4]$ but this identification is not true if we take the C_2-action induced by the duality on $\mathrm{Proj}_{\mathbb{C}}$ into account. The C_2-action on the spectra ko and ku induced by the duality is given by ψ^{-1} and with these actions the fibre sequence becomes a C_2-equivariant fibre sequence. Let us analyse these actions a bit. First of all, the action on ku is given on the homotopy groups of ku by sending the Bott element β to $-\beta$. It is the connective cover of the usual C_2-action on KU which has the property that $\mathrm{KU}^{hC_2} = \mathrm{KO}$. It follows that we have

$$\mathrm{ko} = \tau_{\geq 0}\left(\mathrm{ku}^{hC_2}\right)$$

and that the C_2 action on ko is trivial.

Remark 15.3. The fibre of the map (6) is also called the (2-adic) real image of J-spectrum and denoted by $j_{\mathbb{R}}$ we will also later use the variant

$$j'_{\mathbb{R}} := \mathrm{fib}\left(\mathrm{ko}_2 \xrightarrow{\psi^3-1} \tau_{\geq 2}\mathrm{ko}_2\right)$$

which receives a map $j_{\mathbb{R}} \to j'_{\mathbb{R}}$ that is an isomorphism in degrees > 1. This is why Bökstedt called the fibre of $c(\psi^3-1) : \mathrm{ko}_2 \to \tau_{\geq 4}\mathrm{ku}$ the integral image of J spectrum $j_{\mathbb{Z}}$. There are also non-connective variants of the image of j spectra, e.g. the spectrum

$$J_{\mathbb{R}} = \mathrm{fib}\left(\mathrm{KO}_2 \xrightarrow{\psi^3-1} \mathrm{KO}_2\right)$$

which is equivalent to the $K(1)$-local sphere $\mathbb{S}_{K(1)}$ at the prime 2. Similarly one has the spectrum

$$J_{\mathbb{Z}} = \mathrm{fib}\left(\mathrm{KO}_2 \xrightarrow{c(\psi^3-1)} \mathrm{KU}_2\right)$$

which is the $K(1)$-localization of $K(\mathbb{Z})$ at the prime 2.

We now use the cofibre sequence

$$K(\mathbb{Z})_{hC_2} \to \mathrm{GW}^s(\mathbb{Z}) \to l^s(\mathbb{Z})$$

and the map $\mathrm{GW}^s(\mathbb{Z}) \to K(\mathbb{Z})^{hC_2}$ to get a map $l^s(\mathbb{Z}) \to \mathrm{K}(\mathbb{Z})^{tC_2}$.

Lemma 15.4. *The map* $L^s(\mathbb{Z}) \to K(\mathbb{Z})^{tC_2}$ *is a 2-completion.*

Proof. We first argue that the homotopy groups of both sides are abstractly isomorphic. Recall that the homotopy groups of the source are given by $\mathbb{Z}$ in degrees $*$ mod 4 and $\mathbb{Z}/2$ in degrees $* = 1 \mod 4$. To this end we use the fibre sequence of Theorem 15.2 to get a fibre sequence

$$K(\mathbb{Z})^{tC_2} \to \mathrm{ko}^{tC_2} \to (\tau_{\geq 4}\mathrm{ku})^{tC_2}$$

Now one uses the Tate spectral sequences to deduces that the homotopy groups of ko^{tC_2} are give by $\mathbb{Z}[x^{\pm}]$ for $|x| = 4$ and the homotopy groups of $(\tau_{\geq 4}\mathrm{ku})^{tC_2}$ are given by $\mathbb{Z}/2$ in degrees 2 mod 4. The differentials can all be imported from the HFPSS of C_2-acting by conjugation on KU. We deduce that the homotoy groups of $K\mathbb{Z}^{tC_2}$ are abstractly isomorphic to the ones of $L^s(\mathbb{Z})$. Now we need to show that the map is an isomorphism in all degrees. It suffices to show that it is surjective. For the classes in degree $0 \mod 4$ this follows from the existence of ring structures and for the classes in degree $1 \mod 4$ one has to use a localization seqeunce that we skip here. $\square$

Theorem 15.5. *We have 2-adic equivalences of spectra*

$$\mathrm{GW}^s(\mathbb{Z})_2 \simeq \tau_{\geq 0}K(\mathbb{Z})_2^{hC_2} \simeq j'_{\mathbb{R}} \oplus \mathrm{ko}_2$$

with

$$j'_{\mathbb{R}} := \mathrm{fib}\left(\mathrm{ko}_2 \xrightarrow{\psi^3-1} \tau_{\geq 2}\mathrm{ko}_2\right) .$$

Proof. We use the fibre square to deduce the first result. For the second result we invoke the fibre sequence

$$K(\mathbb{Z})^{hC_2} \to \mathrm{ko}^{hC_2} \to (\tau_{\geq 4}\mathrm{ku})^{hC_2}$$

and observe that since the C_2-action on ko is trivial we get 2-adically by the Atiyah–Segal completion theorem that

$$\tau_{\geq 0}\mathrm{ko}^{hC_2} = \tau_{\geq 0}\mathrm{KO}^{hC_2} = \tau_{\geq 0}\mathrm{KO}^{hC_2} = \tau_{\geq 0}(\mathrm{KO} \oplus \mathrm{KO}) = \mathrm{ko} \oplus \mathrm{ko}$$

Now one uses the HFPSS in the second case to deduce that we have an equivalence

$$\tau_{\geq 0}\left(\tau_{\geq 4}\mathrm{ku}\right)^{hC_2} = \tau_{\geq 2}ko$$

Thus using naturality of the Adams operations we get that the connective cover of $K(\mathbb{Z})^{hC_2}$ is given by the fibre of the map

$$\mathrm{ko} \oplus \mathrm{ko} \xrightarrow{\nabla} \mathrm{ko} \xrightarrow{\psi^3-1} \tau_{\geq 2}\mathrm{ko} .$$

Since the first map is a split surjection we deduce that this fibre is given by

$$\mathrm{fib}(\mathrm{ko} \oplus \mathrm{ko} \xrightarrow{\nabla} \mathrm{ko}) \oplus \mathrm{fib}(\mathrm{ko} \xrightarrow{\psi^3-1} \tau_{\geq 2}\mathrm{ko})$$

which shows the claim. $\square$

Now from the knowledge of the Adams operation ψ^3 which sends the Bott element to 3 times itself and fixes η one can easily compute the homotopy groups of the image of j spectrum (or look it up in the literature). For example one gets that $\pi_3(j'_{\mathbb{R}})$ is given by the cokernel of the map $\psi^3 - 1 : KO_4 \to KO_4$. The generator of $KO_4 = \mathbb{Z}$ is given by $2\beta^2$, more precisely an element that maps to 2β under the complexification map $\mathrm{ko}_4 \to \mathrm{ku}_4 = \mathbb{Z}\{\beta^2\}$. Thus the map ψ^3 acts by multiplication with 9 and therefore $\pi_3(j'_{\mathbb{R}}) = \mathbb{Z}/8$.

Using some results from number theory we arrive at the following computation of $\mathrm{GW}^s_n(\mathbb{Z})$ for $n > 0$ as

$n =$	$\mathrm{GW}^s_n(\mathbb{Z})$
$8k$	$\mathbb{Z} \oplus \mathbb{Z}/2$
$8k+1$	$(\mathbb{Z}/2)^3$
$8k+2$	$(\mathbb{Z}/2)^2 \oplus \mathrm{K}_{8k+2}(\mathbb{Z})_{odd}$
$8k+3$	$\mathbb{Z}/w_{4k+2}$
$8k+4$	$\mathbb{Z}$
$8k+5$	0
$8k+6$	$\mathrm{K}_{8k+6}(\mathbb{Z})_{\mathrm{odd}}$
$8k+7$	$\mathbb{Z}/w_{4k+4}$

where w_{2n} is the denominator of $|\frac{B_{2n}}{4n}|$

Note that the equivalence $\mathrm{GW}^s(\mathbb{Z})_2 \simeq \tau_{\geq 0} K(\mathbb{Z})_2^{hC_2}$ was an open conjecture for quite some time, known as the homotopy limit Problem. One can ask this in much greater generality and much work had been done on that. It had essentially been solved for $R[\frac{1}{2}]$ where R is a ring of integers in a number field by work of several people. With our methods we can now extend this result to the following theorem:

Theorem 15.6. *Let R be a Dedekind ring whose fraction field is a number field. Then the map*

$$\mathrm{GW}^s(R)_2 \to \tau_{\geq 0} K(R)_2^{hC_2}$$

is an equivalence.

The idea is to translate this immediately into an L-theoretic statement which is relatively straighforward to verify as we have demonstrated in the case of the integers.

Finally we also want to compute the quadratic Grothendieck-Witt groups of the integers. We can in fact calculate the cofibre of the map $\mathrm{GW}^q(R) \to \mathrm{GW}^s(R)$ as the cofibre of the connective cover of the map $L^{gq}(\mathbb{Z}) \to L^{sq}(\mathbb{Z})$ using the computations in Theorem 11.10 of both sides. Using this we obtain the following result:

Theorem 15.7. *The quadratic Grothendieck–Witt groups of $\mathbb{Z}$ are given by*

(1) $\mathrm{GW}^q_0(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$
(2) $\mathrm{GW}^q_1(\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$
(3) $\mathrm{GW}^q_*(\mathbb{Z}) = \mathrm{GW}^s_*(\mathbb{Z})$ *for* $* \geq 2$

Remark 15.8. We deduce from Theorem 15.6 and Theorem 15.8 that the $K(1)$-localizations of $\mathrm{GW}^q(\mathbb{Z})$ and $\mathrm{GW}^s(\mathbb{Z})$ are equivalent and given by the $K(1)$-localization of $\mathbb{S} \oplus \mathrm{KO}$.

I find it very surprising that these groups agree in high degree. In fact using the comparisons of L-groups as given in Theorem 11.9 (based on the t-structure surgery from Section 9) one immediately gets the following general result:

Theorem 15.9. *For any Noetherian ring R with involution of global dimension d the canonical map $\mathrm{GW}^q_*(R) \to \mathrm{GW}^s_*(R)$ is injective for $* \geq d+2$ and an isomorphism for $* \geq d+3$.*